On power series solutions for the Euler equation, and the Behr-Nečas-Wu initial datum

Carlo Morosi [a], Mario Pernici [b], Livio Pizzocchero [c] ([1])

[a] Dipartimento di Matematica, Politecnico di Milano,
P.za L. da Vinci 32, I-20133 Milano, Italy
e–mail: carlo.morosi@polimi.it

[b] Istituto Nazionale di Fisica Nucleare, Sezione di Milano,
Via Celoria 16, I-20133 Milano, Italy
e–mail: mario.pernici@mi.infn.it

[c] Dipartimento di Matematica, Università di Milano
Via C. Saldini 50, I-20133 Milano, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy
e–mail: livio.pizzocchero@unimi.it

Abstract

We consider the Euler equation for an incompressible fluid on a three dimensional torus, and the construction of its solution as a power series in time. We point out some general facts on this subject, from convergence issues for the power series to the role of symmetries of the initial datum. We then turn the attention to a paper by Behr, Nečas and Wu [5]; here, the authors chose a very simple Fourier polynomial as an initial datum for the Euler equation and analyzed the power series in time for the solution, determining the first 35 terms by computer algebra. Their calculations suggested for the series a finite convergence radius τ_3 in the H^3 Sobolev space, with $0.32 < \tau_3 < 0.35$; they regarded this as an indication that the solution of the Euler equation blows up.

We have repeated the calculations of [5], using again computer algebra; the order has been increased from 35 to 52, using the symmetries of the initial datum to speed up computations. As for τ_3, our results agree with the original computations of [5] (yielding in fact to conjecture that $0.32 < \tau_3 < 0.33$). Moreover, our analysis supports the following conclusions:
(a) The finiteness of τ_3 is not at all an indication of a possible blow-up.
(b) There is a strong indication that the solution of the Euler equation does not blow up at a time close to τ_3. In fact, the solution is likely to exist, at least, up to a time $\theta_3 > 0.47$.
(c) Padé analysis gives a rather weak indication that the solution might blow up at a later time.

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[1] Corresponding author

1 Introduction

Let us consider the three-dimensional Euler equation for a homogeneous incompressible fluid (of unit density) with initial datum u_0, i.e.,

$$\frac{\partial u}{\partial t} = -u \bullet \nabla u - \nabla p \ , \qquad u(x,0) = u_0(x) \ .$$

The unknown is the divergence free velocity field $(x,t) \mapsto u(x,t)$; we assume periodic boundary conditions, so $x = (x_1, x_2, x_3)$ ranges in the three dimensional torus $(\mathbf{R}/2\pi\mathbf{Z})^3$. In the sequel, we often write $u(t)$ for the function $x \mapsto u(x,t)$.

One can try a solution of the above Cauchy problem in the form of a power series $u(t) = \sum_{j=0}^{+\infty} u_j t^j$ (with $u_j = u_j(x)$); such power series have been the object of rather extensive investigations. Morf *et al* [15], Frisch [12], Brachet *et al* [8], Pelz [20], and other authors (see the bibliography of the cited references) have constructed by computer algebra techniques many terms of the power series for specific initial data, consisting of simple Fourier polynomials; more precisely, the data analyzed in these works are the so-called "Taylor-Green vortex", and other vortices proposed by Kida [14]. The cited authors have also discussed the possibility of a blow-up (i.e., finite-time divergence of $u(t)$) on the grounds of their computer algebra calculations. Another initial datum (again a Fourier polynomial) has been considered by Behr, Nečas and Wu [5]; these authors have constructed 35 terms of the power series, and claimed to have found evidence for a blow-up of the solution; however, in comparison with the vortices of Taylor-Green and Kida, the Behr-Nečas-Wu initial datum has received less attention in the literature.

The purpose of the present paper is twofold.

(i) First of all, we wish to point out a number of general facts on the solutions of the Euler equation and, in particular, on the convergence of the power series $\sum_{j=0}^{+\infty} u_j t^j$; this is the subject of Sections 2 and 3. Here we report some results extracted from the existing literature on the Euler equation in spaces of analytic functions and/or in Sobolev spaces; in addition to these results, we present some remarks of ours and propose a general treatment to discuss the symmetries of the initial datum and their effects on the solution of the Euler equation. We think it is not useless to collect all these theoretical statements in a unifying framework, suitable for direct application to computer algebra calculations.

(ii) Our second aim is to reanalyze the power series for the Behr-Nečas-Wu initial datum, both from the theoretical and from the computational viewpoint; this is the subject of Sections 4, 5 and 6. First of all we apply to the Behr-Nečas-Wu case our general setting for the symmetries of the initial datum. We calculate the symmetry subgroup of the Behr-Nečas-Wu datum (that we recognize to be the dihedral group of order 6; this group also determines what we call the pseudo-symmetry space of the datum).

With these premises, we present a novel computation of the power series for the Behr-Nečas-Wu datum, based on a Python program written for this purpose; this computation attains the order 52. The Python program uses an exact representation of rational numbers as ratios of integer, so as not to introduce rounding errors; furthermore, it employs the symmetries of the initial datum to reduce the amount of calculations.

The results of such computations can be analyzed using the theoretical framework of Sections 2 and 3. Our conclusions are the following:
(a) We agree with the estimates of [5], according to which the power series under consideration has a convergence radius $0.32 < \tau_3 < 0.35$ in the Sobolev space H^3; in fact, our computations suggest $0.32 < \tau_3 < 0.33$. However, we disagree from the authors of [5] when they interpret the finiteness of τ_3 as indicating a blow-up of the solution.
(b) On the contrary, we give evidence that the solution $u(t)$ of the Euler equation exists for t sensibly larger than τ_3. In fact, analyzing the power series for the squared Sobolev norm $\|u(t)\|_3^2$, we find a strong indication for a convergence radius θ_3 such that $0.47 < \theta_3 < 0.50$. By a general criterion *à la* Beale-Kato-Majda, this implies that the solution of the Euler equation exists, at least, up to time θ_3.

The final part of our analysis concerns an alternative approach to estimate θ_3, and the possibility that $u(t)$ blows up at times larger than θ_3. In connection with this problem we use the idea (employed in [8] [12] [15] [20] for different initial data) to construct the Padé approximants for the (squared) Sobolev norms and analyze their singularities. In particular, we construct the diagonal Padé approximants $[p/p](t)$ for $\|u(t)\|_3^2$, up to $p = 26$. For most of them the complex singularities of minimum modulus have modulus $\simeq 0.5$; this fact yields new evidence for the previous estimate on θ_3. Moreover, most of these Padé approximants have real singularities, distributed rather erratically; analyzing them in terms of mean value and variance, we obtain a somehow weak indication that:
(c) $u(t)$ might blow up for $t \to T^-$ (and $t \to (-T)^+$), for some T such that $0.56 < T < 0.73$.

The blow-up problem can be studied as well in terms of D-log Padé approximants; these do not give a clear indication supporting conjecture (c), as briefly explained at the end of the paper. In general, much caution is recommended about the Euler equation and blow-up predictions via Padé analysis: for example, in the case of the Taylor-Green vortex the Padé approximants exhibit real singularities [12] [15], but the numerical solution of the Euler equation by spectral methods raises doubts on the actual existence of a blow-up [7] [9].

Connections with other works. Concluding this Introduction, to put the subject of this paper into a wider perspective we wish to mention that there are general methods of functional analysis to obtain quantitive lower bounds on the time of existence T of the solution of the Euler (or Navier-Stokes) Cauchy problem, from

the *a posteriori* analysis of an approximate solution; such lower bounds are certain (i.e., non conjectural).

Derivations of such *a posteriori* lower bounds have been given in [11] [17] [19]. The last of these works gives an algorithm to obtain these lower bounds analyzing any approximate solution of the Euler (or Navier-Stokes) Cauchy problem via a suitable differential inequality, called therein the "control inequality".

Again in [19], a preliminary analysis of the Euler (and Navier-Stokes) equations with the Behr-Nečas-Wu initial datum has been performed, using for the solution a Galerkin approximation with very few Fourier modes. This approximant, combined with the control inequality, gives for the Euler equation with this datum a (poor, but certain) lower bound $T > 0.066$ for the time of existence in H^3 (the same approach, applied to the Navier-Stokes equations, grants $T = +\infty$ when the viscosity coefficient is above an explicit threshold). We plan to continue in future works the analysis of the Behr-Nečas-Wu intial datum, combining the control equation of [19] with approximation methods based on extensive automatic computations such as the ones presented in this paper.

2 The Cauchy problem for the Euler equation on a torus

Preliminaries. If $a = (a_s), b = (b_s)$ are elements of $\mathbf{R}^3$ or $\mathbf{C}^3$, we intend $a \bullet b := \sum_{s=1}^3 a_s b_s$. We indicate with $\bar{\ }$ the complex conjugate (and we let it act componentwise on elements of $\mathbf{C}^3$); we put $|a| := \sqrt{\bar{a} \bullet a} = \sqrt{\sum_{s=1}^3 |a_s|^2}$.

The Cauchy problem for the incompressible Euler equation is

$$\frac{\partial u}{\partial t} = -u \bullet \nabla u - \nabla p \ , \qquad u(x, 0) = u_0(x) \ , \tag{2.1}$$

where: $u = u(x,t)$ is the divergence free velocity field; the space variables $x = (x_s)_{s=1,2,3}$ belong to the torus $\mathbf{T}^3 := (\mathbf{R}/2\pi\mathbf{Z})^3$; $(u \bullet \nabla u)_r := \sum_{s=1}^3 u_s \partial_s u_r$ $(r = 1, 2, 3)$; $p = p(x,t)$ is the pressure; $u_0 = u_0(x)$ is the initial datum. As well known, the pressure can be eliminated from (2.1) using the Leray projection $\mathfrak{L}$ onto the space of divergence free vector fields; this allows to rewrite the evolution equation in (2.1) as $\partial u / \partial t = -\mathfrak{L}(u \bullet \nabla u)$. In this way, we obtain for the Cauchy problem the final form

$$\frac{\partial u}{\partial t} = \mathcal{P}(u, u) \ , \qquad u(\, . \, , 0) = u_0 \ , \tag{2.2}$$

where we have written $\mathcal{P}$ for the bilinear map sending two (sufficiently regular) vector fields $v, w : \mathbf{T}^3 \to \mathbf{R}^3$ into the vector field

$$\mathcal{P}(v, w) := -\mathfrak{L}(v \bullet \nabla w) \ . \tag{2.3}$$

In this framework, it is convenient to associate to a vector field $v : \mathbf{T}^3 \to \mathbf{R}^3$ the Fourier components $v_k := (2\pi)^{-3} \int_{\mathbf{T}^3} dx\, e^{-ik\bullet x} v(x) \in \mathbf{C}^3$, so that

$$v(x) = \sum_{k \in \mathbf{Z}^3} v_k e^{ik\bullet x} \ . \tag{2.4}$$

Due to the reality of v, we have $v_{-k} = \overline{v_k}$, and v is divergence free iff $k\bullet v_k = 0$ for all k. With $\mathcal{P}$ as above and v, w two vector fields, the Fourier components of $\mathcal{P}(v, w)$ are

$$\mathcal{P}(v, w)_k = -i \sum_{h \in \mathbf{Z}^3} v_h \bullet (k - h)\, \mathfrak{L}_k w_{k-h} \tag{2.5}$$

where $\mathfrak{L}_k : \mathbf{C}^3 \to \mathbf{C}^3$ is the projection on the orthogonal complement of k ($\mathfrak{L}_k c := c - (k\bullet c)k/|k|^2$ if $k \neq 0$; $\mathfrak{L}_0 c := c$).

In the above, we have introduced the setting for the Euler equation in an informal way; to go on, it is necessary to specify the functional spaces to which the velocity fields (at any time) are supposed to belong.

The expression "a vector field $\mathbf{T}^3 \to \mathbf{R}^3$" can be understood, with very wide generality, as "an $\mathbf{R}^3$-valued distribution on $\mathbf{T}^3$" (see, e.g., [18]); we write $D'(\mathbf{T}^3, \mathbf{R}^3) \equiv D'$ for the space of such distributions. Any $v \in D'(\mathbf{T}^3, \mathbf{R}^3)$ can be differentiated in the distributional sense and has a (weakly convergent) Fourier expansion with coefficients $v_k \in \mathbf{C}^3$, such that $\overline{v_k} = v_{-k}$.

To construct the full setting for the Euler equation, one must confine the attention to much smaller functional spaces of vector fields. For our purposes, two cases are important:

(i) The Sobolev space H^n of zero mean, divergence free vector fields of any order $n \in [0, +\infty)$. This is defined in terms of the space $L^2(\mathbf{T}^3, \mathbf{R}^3) \equiv L^2$ of square integrable vector fields $v : \mathbf{T}^3 \to \mathbf{R}^3$, equipped with the inner product $\langle v|w\rangle_{L^2} := (2\pi)^{-3} \int_{\mathbf{T}^3} v(x)\bullet w(x) dx$ and with the induced norm $\|v\|_{L^2} = (2\pi)^{-3/2} \sqrt{\int_{\mathbf{T}^3} |v(x)|^2 dx}$ (note the term $(2\pi)^{-3}$ in the inner product, used systematically in the sequel). By definition,

$$H^n(\mathbf{T}^3, \mathbf{R}^3) \equiv H^n := \Big\{ v \in D' \mid \sqrt{-\Delta}^n v \in L^2 \ , \int_{\mathbf{T}^3} v \ dx = 0, \ \operatorname{div} v = 0 \Big\} \tag{2.6}$$

$$= \Big\{ v \in D' \mid \sum_{k \in \mathbf{Z}^3} |k|^{2n} |v_k|^2 < +\infty, v_0 = 0, k\bullet v_k = 0 \Big\} \ .$$

(In the above $\sqrt{-\Delta}^n$ indicates the power of order $n/2$ of minus the Laplacian; by definition $(\sqrt{-\Delta}^n v)_k = |k|^n v_k$ for each $v \in D'$. Note that $H^n \subset L^2$ for all $n \geqslant 0$.) H^n is a Hilbert space with the inner product

$$\langle v|w\rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w\rangle_{L^2} = \sum_{k \in \mathbf{Z}^3} |k|^{2n} \overline{v_k} \bullet w_k \ , \tag{2.7}$$

inducing the norm

$$\|v\|_n = \|\sqrt{-\Delta}^n v\|_{L^2} = \sqrt{\sum_{k \in \mathbf{Z}^3} |k|^{2n} |v_k|^2} \ . \tag{2.8}$$

It is known that $\mathcal{P}$ sends continuously $H^n \times H^{n+1}$ into H^n, for all $n \in (3/2, +\infty)$.
(ii) The space of C^ω (i.e., analytic) zero mean, divergence free vector fields on $\mathbf{T}^3$; this is

$$\mathcal{A}(\mathbf{T}^3, \mathbf{R}^3) \equiv \mathcal{A} := \left\{ v \in C^\omega(\mathbf{T}^3, \mathbf{R}^3) \ \Big| \ \int_{\mathbf{T}^3} v \ dx = 0, \ \operatorname{div} v = 0 \ \right\} \tag{2.9}$$

$$= \left\{ v \in D' \ \Big| \ \liminf_{k \in \mathbf{Z}^3, \, k \to \infty} |v_k|^{-\frac{1}{|k_1|+|k_2|+|k_3|}} > 1, v_0 = 0, \ k \bullet v_k = 0 \right\}$$

(intending $0^{-\frac{1}{|k_1|+|k_2|+|k_3|}} := +\infty$. The Fourier representation in (2.9) mimics the description of analytic functions on the torus in [16], which is also a useful reference for what follows). One has

$$\mathcal{A} = \cup_{\rho \in (1, +\infty)} \mathcal{A}_\rho \ , \tag{2.10}$$

$$\mathcal{A}_\rho := \left\{ v \in D' \ \Big| \ \liminf_{k \in \mathbf{Z}^3, \, k \to \infty} |v_k|^{-\frac{1}{|k_1|+|k_2|+|k_3|}} > \rho, \ v_0 = 0, \ k \bullet v_k = 0 \right\} ;$$

each $\mathcal{A}_\rho$ is a vector subspace of $\mathcal{A}$. Let us introduce the annulus $K_\rho := \{z \in \mathbf{C} \mid 1/\rho \leqslant |z| \leqslant \rho\}$ and its power $K^3_\rho := \{z = (z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_1, z_2, z_3 \in K_\rho\}$. For $v \in \mathcal{A}_\rho$, the series $\sum_{k \in \mathbf{Z}^3} v_k z^k$ converges in $\mathbf{C}^3$ for each $z \in K^3_\rho$ (intending $z^k := z_1^{k_1} z_2^{k_2} z_3^{k_3}$); the function $z \mapsto \sum_{k \in \mathbf{Z}^3} v_k z^k$ is holomorphic on the inner part of K^3_ρ and continuous on K^3_ρ, so we can define

$$||| \, v |||_\rho := \sup_{z \in K^3_\rho} \left| \sum_{k \in \mathbf{Z}^3} v_k z^k \right| \ . \tag{2.11}$$

$||| \ |||_\rho$ is a norm on $\mathcal{A}_\rho$ and makes it a Banach space. One equips $\mathcal{A}$ with the inductive limit topology of the collection of Banach spaces $\{(\mathcal{A}_\rho, ||| \ |||_\rho) \mid \rho \in (1, +\infty)\}$: this is the finest locally convex topology on $\mathcal{A}$ making continuous each embedding $\mathcal{A}_\rho \hookrightarrow \mathcal{A}$. (Besides [16], see [23] for the general theory of inductive limits.) $\mathcal{A}$ is continuously embedded into each Sobolev space H^n; the map $\mathcal{P}$ is continuous from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$.

Basic results on local existence and uniqueness. We start from the Sobolev framework, choosing

$$n \in (5/2, +\infty) \ . \tag{2.12}$$

In the sequel, an H^n-solution of the Euler equation, or of the Euler Cauchy problem, means a map

$$u \in C((-\mathcal{T}, T), H^n) \cap C^1((-\mathcal{T}, T), H^{n-1}) \tag{2.13}$$

($\mathcal{T}, T \in (0, +\infty]$) fulfilling the Euler equation, or its Cauchy problem with a suitable initial condition $u(0) = u_0$. The following statement is well known:

2.1 ***Proposition.*** *For n as above and any initial datum $u_0 \in H^n$, the following holds.*
(i) The Cauchy problem (2.2) has a unique maximal (i.e., not extendable) H^n-solution u of domain $(-\mathcal{T}, T)$, for suitable $T = T(u_0), \mathcal{T} = \mathcal{T}(u_0) \in (0, +\infty]$.
(ii) (Beale-Kato-Majda criterion, Sobolev version). If $T < +\infty$, one has

$$\int_0^T dt\, \|u(t)\|_n = +\infty \ , \tag{2.14}$$

a fact implying

$$\limsup_{t \to T^-} \|u(t)\|_n = +\infty \ . \tag{2.15}$$

Similar results hold if $\mathcal{T} < +\infty$, considering the integral from $-\mathcal{T}$ to 0 and the limit for $t \to (-\mathcal{T})^+$.

Proof. (i) See [4] [13].
(ii) See [4]. Indeed, here it is shown that $T < +\infty$ implies $\int_0^T dt \|\mathrm{rot}\, u(t)\|_{L^\infty} = +\infty$; however, $\|\mathrm{rot}\, u(t)\|_{L^\infty} \leqslant \text{const.} \|u(t)\|_n$ by the Sobolev imbedding inequalities, whence Eq. (2.14) and its obvious consequence (2.15). The behavior of u at time $-\mathcal{T}$ is analyzed similarly. $\square$

If $T < +\infty$, the solution u is said to blow up at time T. Similarly, if $\mathcal{T} < +\infty$ we say that u blows up at $-\mathcal{T}$. Many statements presented in the sequel on the possibility of blow-up at T have obvious reformulations regarding $-\mathcal{T}$.

2.2 Remark. The Beale-Kato-Majda criterion (2.14) yields the following statement, in case of blow-up with a power law:

$$\text{if } \|u(t)\|_n \sim \frac{U}{(T-t)^\alpha} \quad \text{for } t \to T^- \text{ (with } U, \alpha > 0), \quad \text{then } \alpha \geqslant 1 \ . \tag{2.16}$$

In the case of the Euler equation on $\mathbf{R}^3$, it was recently shown in [10], Theorem 1.3 that the blow-up at T implies the following, for any $n > 5/2$:

$$\|u(t)\|_n \geqslant \frac{U}{(T-t)^{2n/5}} \quad \text{for } t \text{ close to } T \ , \qquad U = U_n(\|u_0\|_{L^2}) \ . \tag{2.17}$$

This estimate might hold as well for the framework of the present paper, i.e., for the Euler equation on the torus $\mathbf{T}^3$ (however, the extendability of (2.17) to $\mathbf{T}^3$ is immaterial for the purposes of this paper). $\square$

Let us pass to the C^ω (= analytic) framework; what follows assumes some general notions from the theory of analytic functions from $\mathbf{R}$ to locally convex spaces, for which we refer to [6] §3. Let $\mathcal{A}$ be the space (2.9); in the sequel, an $\mathcal{A}$-solution of the Euler equation, or of the Euler Cauchy problem, means a map

$$u \in C^\omega((-\mathcal{T}, T), \mathcal{A}) \tag{2.18}$$

($\mathcal{T}, T \in (0, +\infty]$) fulfilling the Euler equation, or its Cauchy problem with a suitable initial condition $u(0) = u_0$. Let us report a known result.

2.3 ***Proposition.*** *For any initial datum $u_0 \in \mathcal{A}$, the following holds.*
(i) Problem (2.2) has a unique maximal (i.e., non extendable) $\mathcal{A}$-solution of domain $(-\mathcal{T}, T)$, for suitable $T = T(u_0), \mathcal{T} = \mathcal{T}(u_0) \in (0, +\infty]$.
(ii) For any $n \in (5/2, +\infty)$, this coincides with the maximal H^n-solution of the Cauchy problem with the same datum (and thus, if $T < +\infty$, it fulfills Eqs. (2.14) (2.15); a similar result holds if $\mathcal{T} < +\infty$).

Proof. (i) See [2], Theorem III.2, page 264 (this is a result of existence and uniqueness on sufficiently small time intervals, from which one infers via standard arguments existence and uniqueness of the maximal solution).
(ii) See [3], especially Remark 2.1, page 414. □

Assuming again $u_0 \in \mathcal{A}$, and choosing any $n \in [0, +\infty)$, we conclude with two remarks.
(i) By the continuous embedding of $\mathcal{A}$ into H^n, the function u of the last proposition is also in $C^\omega((-\mathcal{T}, T), H^n)$.
(ii) Consider the function

$$(-\mathcal{T}, T) \to \mathbf{R} , \qquad t \mapsto \|u(t)\|_n^2 . \tag{2.19}$$

This is in $C^\omega((-\mathcal{T}, T), \mathbf{R})$, being the composition of the analytic function $u : (-\mathcal{T}, T) \to H^n$ with the continuous quadratic function $\| \ \|_n^2 : H^n \to \mathbf{R}$.

Symmetries of the Euler equation. Let us consider the *octahedral group* O_h, formed by the orthogonal 3×3 matrices with integer entries:

$$O_h := \{ S \in Mat(3 \times 3, \mathbf{Z}) \mid S^{\mathrm{T}} S = \mathbf{1}_3 \} . \tag{2.20}$$

In fact, the entries of any such matrix have $-1, 0$ and 1 as the only possible values; furthermore, a 3×3 matrix S belongs to O_h if and only if

$$S = \mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3) Q(\sigma) \tag{2.21}$$

$\epsilon_s \in \{\pm 1\}$ $(s = 1, 2, 3)$; $Q(\sigma)$ the matrix of the permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$;

more precisely, $Q(\sigma)$ is the matrix such that $(Q(\sigma)c)_s = c_{\sigma(s)}$ for all $c \in \mathbf{C}^3$, $s \in$

$\{1, 2, 3\}$. There are $2^3 = 8$ possible choices for the signs ϵ_i and $3! = 6$ choices for σ, so O_h has $8 \times 6 = 48$ elements. Clearly, each $S \in O_h$ sends $\mathbf{Z}^3$ into itself.

To go on, let us denote with $O_h \ltimes \mathbf{T}^3$ the Cartesian product $O_h \times \mathbf{T}^3$, viewed as a group with the composition law defined by ([2])

$$(S, a)(U, b) := (SU, a + Sb) \quad (S, U \in O_h \ ; \ a, b \in \mathbf{T}^3) \ . \tag{2.22}$$

Of course, the unit of this group is $(\mathbf{1}, 0)$ (with $\mathbf{1}$ the identity 3×3 matrix); the inverse of a pair (S, a) is $(S, a)^{-1} = (S^{\mathrm{T}}, -S^{\mathrm{T}} a)$. To any element (S, a) of $O_h \ltimes \mathbf{T}^3$ is associated a "rototranslation"

$$\mathcal{E}(S, a) : \mathbf{T}^3 \to \mathbf{T}^3 \ , \qquad x \mapsto \mathcal{E}(S, a)(x) := Sx + a \ , \tag{2.23}$$

and one checks that the mapping $(S, a) \mapsto \mathcal{E}(S, a)$ is a group homomorphism between $O_h \ltimes \mathbf{T}^3$ and the group of diffeomorphisms of $\mathbf{T}^3$ into itself (with the usual composition).

Now, we take a vector field v in H^n (or in $\mathcal{A}$) and an element (S, a) of the group $O_h \ltimes \mathbf{T}^3$. We can construct the push-forward $\mathcal{E}_*(S, a)v$ of v along the mapping $\mathcal{E}(S, a)$; this is the vector field in H^n (or in $\mathcal{A}$), given by

$$\mathcal{E}_*(S, a)v : \mathbf{T}^3 \to \mathbf{R}^3 \ , \qquad x \mapsto (\mathcal{E}_*(S, a)\, v)(x) = Sv(S^{\mathrm{T}}(x - a)) \ . \tag{2.24}$$

One easily checks that Eq. (2.24) actually defines a vector field in H^n (or in $\mathcal{A}$), with Fourier components

$$(\mathcal{E}_*(S, a)\, v)_k = e^{-i a \bullet k} S v_{S^{\mathrm{T}} k} \qquad (k \in \mathbf{Z}^3) \ . \tag{2.25}$$

Let us write $\mathcal{E}_*(S, a)$ for the map $v \in H^n \mapsto \mathcal{E}_*(S, a)v$; this is a linear map of H^n into itself, preserving the inner product $\langle \ | \ \rangle_n$, so it is in the group $O(H^n)$ of orthogonal operators of the Hilbert space H^n into itself. The mapping

$$\mathcal{E}_* : O_h \ltimes \mathbf{T}^3 \to O(H^n) \ , \qquad (S, a) \mapsto \mathcal{E}_*(S, a) \tag{2.26}$$

is a injective group homomorphism, i.e., a faithful orthogonal representation of the group $O_h \ltimes \mathbf{T}^3$ on the real Hilbert space H^n. Alternatively, let us write $\mathcal{E}_*(S, a)$ for the map $v \in \mathcal{A} \mapsto \mathcal{E}_*(S, a)v$; this is in the space $\mathrm{Iso}(\mathcal{A})$ of linear and topological isomorphisms of $\mathcal{A}$ into itself. The map

$$\mathcal{E}_* : O_h \ltimes \mathbf{T}^3 \to \mathrm{Iso}(\mathcal{A}) \ , \qquad (S, a) \mapsto \mathcal{E}_*(S, a) \tag{2.27}$$

is an injective group homomorphism, i.e., a faithful linear representation of the group $O_h \ltimes \mathbf{T}^3$ on the topological vector space $\mathcal{A}$.

[2] This is the semidirect product of the groups O_h and $\mathbf{T}^3$ with respect to the natural homomorphism $O_h \to Aut(\mathbf{T}^3)$ sending $S \in O_h$ into the map $b \mapsto Sb$, an automorphism of $\mathbf{T}^3$.

Let us relate the previous constructions to the bilinear map $\mathcal{P}$ of the Euler equation. From the Fourier representations (2.5) (2.25), one easily infers

$$\mathcal{P}(\mathcal{E}_*(S,a)\,v, \mathcal{E}_*(S,a)\,w) = \mathcal{E}_*(S,a)\,\mathcal{P}(v,w) \tag{2.28}$$

for all $v \in H^n$, $w \in H^{n+1}$ with $n > 3/2$ (and, in particular, for all $v, w \in \mathcal{A}$). Let us outline the implications of (2.28) about the solutions of the Euler equation. In the rest of the paragraph, the term "solution" either means an H^n-solution ($n > 5/2$) or an $\mathcal{A}$-solution, and the initial datum u_0 is chosen consistently in H^n or in $\mathcal{A}$. From (2.28) one infers the following, for each $(S,a) \in O_h \ltimes \mathbf{T}^3$:
(i) If $u : t \in (-\mathcal{T}, T) \mapsto u(t)$ is a solution of the Euler equation, we have two more solutions

$$\mathcal{E}_*(S,a)u : \; t \in (-\mathcal{T}, T) \mapsto \mathcal{E}_*(S,a)u(t), \tag{2.29}$$

$$-\mathcal{E}_*(S,a)u(-\cdot) : \; t \in (-T, \mathcal{T}) \mapsto -\mathcal{E}_*(S,a)u(-t). \tag{2.30}$$

(ii) If $u : t \in (-\mathcal{T}, T) \mapsto u(t)$ is the maximal solution of the Euler Cauchy problem with datum u_0, then $\mathcal{E}_*(S,a)u$ is the maximal solution with datum $\mathcal{E}_*(S,a)u_0$ and $-\mathcal{E}_*(S,a)u(-\cdot)$ is the maximal solution with datum $-\mathcal{E}_*(S,a)u_0$.
(iii) Let us denote again with $u : t \in (-\mathcal{T}, T) \mapsto u(t)$ the maximal solution of the Cauchy problem with datum u_0. Then,

$$\mathcal{E}_*(S,a)u_0 = u_0 \;\Rightarrow\; \mathcal{E}_*(S,a)u(t) = u(t) \qquad \text{for } t \in (-\mathcal{T}, T). \tag{2.31}$$

$$-\mathcal{E}_*(S,a)u_0 = u_0 \;\Rightarrow\; \mathcal{T} = T, \; -\mathcal{E}_*(S,a)u(-t) = u(t) \qquad \text{for } t \in (-T, T). \tag{2.32}$$

The verification of statements (i)(ii) is straightforward. After this, the implication (2.31) in (iii) follows noting that $\mathcal{E}_*(S,a)u$ and u are maximal solutions of the Cauchy problem with the same datum $\mathcal{E}_*(S,a)u_0 = u_0$. Similarly, the implication (2.32) follows noting that $-\mathcal{E}_*(S,a)u(-\cdot)$ and u are maximal solutions of the Cauchy problem with the same datum $-\mathcal{E}_*(S,a)u_0 = u_0$.

Considering the maximal solution u for a datum u_0 in H^n ($n > 5/2$), and recalling that any transformation $\mathcal{E}_*(S,a)$ preserves the H^n norm, we also obtain from (2.32) the following:

$$-\mathcal{E}_*(S,a)u_0 = u_0 \;\Rightarrow\; \mathcal{T} = T, \; \|u(-t)\|_n = \|u(t)\|_n \qquad \text{for } t \in (-T, T). \tag{2.33}$$

The results in (iii) suggest to consider, for a given datum u_0 in H^n or $\mathcal{A}$, the *symmetry subgroup*

$$\mathcal{H}(u_0) := \{(S,a) \in O_h \ltimes \mathbf{T}^3 \mid \mathcal{E}_*(S,a)u_0 = u_0\} \tag{2.34}$$

and the *pseudo-symmetry space*

$$\mathcal{H}^-(u_0) := \{(S,a) \in O_h \ltimes \mathbf{T}^3 \mid -\mathcal{E}_*(S,a)u_0 = u_0\} \tag{2.35}$$

(the first one, being a subgroup of $O_h \ltimes \mathbf{T}^3$, contains at least the identity element $(\mathbf{1}, 0)$; the second one might be the empty set. The term "isotropy group", often employed in place of "symmetry group", will not be used in this paper).
Let us consider the maximal solution u of the Cauchy problem with a datum u_0 (contained in H^n for some $n > 5/2$); from Eqs. (2.31)-(2.33), we readily obtain the following:

$$\mathcal{E}_*(S,a)u(t) = u(t) \qquad \text{for all } (S,a) \in \mathcal{H}(u_0), t \in (-\mathcal{T}, T) \ ; \tag{2.36}$$

$$\mathcal{H}^-(u_0) \neq \emptyset \quad \Rightarrow \quad \mathcal{T} = T, \ -\mathcal{E}_*(S,a)u(-t) = u(t), \ \|u(-t)\|_n = \|u(t)\|_n \tag{2.37}$$
$$\text{for } (S,a) \in \mathcal{H}^-(u_0), t \in (-T, T) \ .$$

For future use, let us introduce the *reduced symmetry subgroup* and the *reduced pseudo-symmetry space* of the datum u_0, which are

$$\mathcal{H}_R(u_0) := \{S \in O_h \mid \mathcal{E}_*(S,a)u_0 = u_0 \text{ for some } a \in \mathbf{T}^3\} \ , \tag{2.38}$$

$$\mathcal{H}_R^-(u_0) := \{S \in O_h \mid -\mathcal{E}_*(S,a)u_0 = u_0 \text{ for some } a \in \mathbf{T}^3\} \ . \tag{2.39}$$

Let us observe that the set theoretical unions $\mathcal{H}(u_0) \cup \mathcal{H}^-(u_0)$ and $\mathcal{H}_R(u_0) \cup \mathcal{H}_R^-(u_0)$ are subgroups of $O_h \ltimes \mathbf{T}^3$ and O_h, respectively.

As a final remark, useful for the sequel, let us consider the pair $(-\mathbf{1}, 0) \in O_h \ltimes \mathbf{T}^3$, noting that $\mathcal{E}(-\mathbf{1}, 0)$ is the space reflection: $\mathcal{E}(-\mathbf{1}, 0)(x) = -x$ for all $x \in \mathbf{T}^3$. One easily checks that

$$(-\mathbf{1}, 0) \in \mathcal{H}^-(u_0) \ \Leftrightarrow \ \mathcal{H}^-(u_0) = \mathcal{H}(u_0)(-\mathbf{1}, 0) = \{(-S, a) \mid (S,a) \in \mathcal{H}(u_0)\} \tag{2.40}$$

(where $\mathcal{H}(u_0)(-\mathbf{1}, 0)$ stands for the set $\{(S,a)(-\mathbf{1}, 0) \mid (S,a) \in \mathcal{H}(u_0)\}$; the last equality rests on the identity $(S,a)(-\mathbf{1}, 0) = (-S, a)$).

3 Power series in time for the Euler Cauchy problem

Throughout this section, we consider the Euler Cauchy problem with initial datum $u_0 \in \mathcal{A}$.

Setting up a power series for the solution. Let us try to build the solution of the Euler Cauchy problem as a power series

$$t \mapsto \sum_{j=0}^{\infty} u_j t^j \tag{3.1}$$

with coefficients $u_j \in \mathcal{A}$, whose convergence has to be discussed later. The zero order term in this expansion is the initial datum u_0; to determine the other coefficients

$u_j \in \mathcal{A}$, it suffices to substitute the expansion (3.1) into the Euler equation (2.2), and to require equality of the coefficients of the same powers of t in both sides: in this way, one easily obtains the recurrence relation

$$u_j = \frac{1}{j} \sum_{\ell=0}^{j-1} \mathcal{P}(u_\ell, u_{j-\ell-1}) \qquad (j = 1, 2, 3, ...) \ . \tag{3.2}$$

When applying this recurrence relation for the u_j's it can be useful to represent the bilinear map $\mathcal{P}$ in terms of Fourier coefficients, as in Eq. (2.5). This is especially useful if the initial datum u_0 is a Fourier polynomial, i.e., if $u_{0k} \neq 0$ only for finitely many modes k. In this case, all the iterates u_j ($j = 1, 2, 3, ..$) are as well Fourier polynomials, and the implementation of (3.2) via the Fourier representation (2.5) always involves sums over finitely many modes.

In the next section, a large part of our attention will be devoted (for a specific datum u_0) to the partial sums

$$u^{(N)}(t) := \sum_{j=0}^{N} u_j t^j \ , \tag{3.3}$$

($N = 0, 1, 2, ...$) and to the (squared) Sobolev norms

$$\| u^{(N)}(t) \|_n^2 = \sum_{k \in \mathbf{Z}^3} |k|^{2n} |u_k^{(N)}(t)|^2 \ . \tag{3.4}$$

Symmetry considerations. Let us consider the symmetry subgroup $\mathcal{H}(u_0)$ or the pseudo-symmetry space $\mathcal{H}^-(u_0)$, see Eqs. (2.34) (2.35). Using the recursive definition (3.2) of u_j with the invariance property (2.28) of $\mathcal{P}$, one easily checks the following, for any $j \in \{0, 1, 2, ...\}$:

$$\mathcal{E}_*(S, a) u_j = u_j \qquad \text{for all } (S, a) \in \mathcal{H}(u_0); \tag{3.5}$$

$$- \mathcal{E}_*(S, a) u_j = (-1)^j u_j \qquad \text{for all } (S, a) \in \mathcal{H}^-(u_0). \tag{3.6}$$

Of course, the last two equations imply the following, for all $N \in \{0, 1, 2, ...\}$, $t \in \mathbf{R}$ and $n \in [0, +\infty)$:

$$\mathcal{E}_*(S, a) u_N(t) = u_N(t) \qquad \text{for } (S, a) \in \mathcal{H}(u_0); \tag{3.7}$$

$$- \mathcal{E}_*(S, a) u_N(t) = u_N(-t) \qquad \text{for } (S, a) \in \mathcal{H}^-(u_0); \tag{3.8}$$

$$\| u_N(t) \|_n = \| u_N(-t) \|_n \qquad \text{if } \mathcal{H}^-(u_0) \neq \emptyset \tag{3.9}$$

(Eq. (3.9) is a consequence of Eq. (3.8) and of the invariance of $\| \ \|_n$ under the transformation $\mathcal{E}_*(S, a)$).

Due to the Fourier representation (2.25) for $\mathcal{E}_*(S,a)$, the equality (3.5) reads $e^{-ia \bullet k}\ Su_{j,S^{\mathsf{T}}k} = u_{j,k}$ or, equivalently,

$$u_{j,Sk} = e^{-ia \bullet Sk} S u_{j,k} \qquad \text{for } k \in \mathbf{Z}^3,\ (S,a) \in \mathcal{H}(u_0)\ ; \tag{3.10}$$

similarly, Eq. (3.6) is equivalent to the statement

$$u_{j,Sk} = (-1)^{j+1} e^{-ia \bullet Sk} S u_{j,k} \qquad \text{for } k \in \mathbf{Z}^3,\ (S,a) \in \mathcal{H}^-(u_0)\ . \tag{3.11}$$

In typical applications of the recursion scheme (3.2), where u_0 is a Fourier polynomial as well as its iterates u_j, Eqs. (3.10) (3.11) can be used to speed up the computation of the Fourier components of the u_j's; in fact, at any given order j, after computing a Fourier component $u_{j,k}$ we immediately obtain from the cited equations the components $u_{j,Sk}$ for all S in the reduced subgroup or subspace $\mathcal{H}_R(u_0)$, $\mathcal{H}_R^-(u_0)$.

Convergence of the power series in $\boldsymbol{\mathcal{A}}$. From now on, we intend

$$\tau := \text{convergence radius of the series } \textstyle\sum_{j=0}^{\infty} u_j t^j \text{ in } \mathcal{A}\ . \tag{3.12}$$

Furthermore,

$$u : t \in (-\mathcal{T}, T) \mapsto u(t) \text{ is the maximal } \mathcal{A}\text{-solution of the Cauchy problem} \tag{3.13}$$

(recall that, for any $n > 5/2$, u is also the maximal H^n-solution). We note that

$$0 < \tau \leqslant \mathcal{T} \wedge T\ , \qquad u(t) = \sum_{j=0}^{\infty} u_j t^j \quad \text{in } \mathcal{A}, \text{ for } t \in (-\tau, \tau) \tag{3.14}$$

(with $\wedge$ indicating the minimum). In fact: being analytic, u admits a power series representation in a neighborhood of zero; this necessarily coincides with the series (3.1), whose convergence radius τ is thus nonzero and fulfills $(-\tau,\tau) \subset (-\mathcal{T}, T)$.

Convergence of the power series in $\boldsymbol{H^n}$. After fixing $n \in [0,+\infty)$, let us discuss the series (3.1) in the Sobolev space H^n. To this purpose, we put

$$\tau_n := \text{convergence radius of the series } \textstyle\sum_{j=0}^{\infty} u_j t^j \text{ in } H^n\ ; \tag{3.15}$$

the root test gives

$$\tau_n = \liminf_{j \to +\infty} \| u_j \|_n^{-1/j} \tag{3.16}$$

(intending $0^{-1/j} := +\infty$). With $\tau, \mathcal{T}, T, u$ as before, we claim that

$$\tau \leqslant \tau_n \text{ and } u(t) = \sum_{j=0}^{\infty} u_j t^j \text{ in } H^n, \text{ for } t \in (-\mathcal{T} \wedge \tau_n, T \wedge \tau_n) \tag{3.17}$$

(where $-\mathfrak{T}\wedge\tau_n$ is the opposite of the minimum $\mathfrak{T}\wedge\tau_n$). In fact: the series $\sum_{j=0}^{\infty} u_j t^j$ converges to $u(t)$ in $\mathcal{A}$, for $t \in (-\tau, \tau)$; by the continuous embedding $\mathcal{A} \hookrightarrow H^n$ this series converges to $u(t)$ in H^n as well, at least for $t \in (-\tau, \tau)$; thus $\tau \leqslant \tau_n$. Moreover the functions $u : (-\mathfrak{T}, T) \to H^n$ and $t \in (-\tau_n, \tau_n) \mapsto \sum_{j=0}^{\infty} u_j t^j \in H^n$ are analytic and coincide on $(-\tau, \tau)$; so, by the analytic continuation principle, these functions coincide on the intersection of their domains which is $(-\mathfrak{T}\wedge\tau_n, T\wedge\tau_n)$. Let as add a stronger claim:

$$\text{if } n > \frac{5}{2}, \qquad \tau \leqslant \tau_n \leqslant \mathfrak{T}\wedge T \text{ and } u(t) = \sum_{j=0}^{\infty} u_j t^j \text{ in } H^n, \text{ for } t \in (-\tau_n, \tau_n)\ . \tag{3.18}$$

In fact, the function $t \in (-\tau_n, \tau_n) \mapsto \sum_{j=0}^{\infty} u_j t^j$ is in $C((-\tau_n, \tau_n), H^n) \cap C^1((-\tau_n, \tau_n), H^{n-1})$ and solves the Euler Cauchy problem, so it is a restriction of the maximal H^n-solution, which is u of domain $(-\mathfrak{T}, T)$; this gives the relations $\tau_n \leqslant \mathfrak{T}\wedge T$ and $u(t) = \sum_{j=0}^{\infty} u_j t^j$ in H^n, for $t \in (-\tau_n, \tau_n)$.

Power series for the Sobolev norms of the solution. Let us choose $n \in [0, +\infty)$. The squared norm $\| \sum_{j=0}^{+\infty} u_j t^j \|_n^2 = \langle \sum_{j=0}^{+\infty} u_j t^j | \sum_{j=0}^{+\infty} u_j t^j \rangle_n$ has the formal expansion

$$\| \sum_{j=0}^{+\infty} u_j t^j \|_n^2 = \sum_{j=0}^{+\infty} \nu_{nj} t^j, \qquad \nu_{nj} := \sum_{\ell=0}^{j} \langle u_\ell | u_{j-\ell} \rangle_n \in \mathbf{R}\ ; \tag{3.19}$$

for future use we remark that ([3])

$$\mathcal{H}^-(u_0) \neq \emptyset \quad \Rightarrow \quad \nu_{nj} = 0 \text{ for all } j \text{ odd}. \tag{3.20}$$

Independently of any assumption on $\mathcal{H}^-(u_0)$, let us define

$$\theta_n := \text{convergence radius of the series } \textstyle\sum_{j=0}^{\infty} \nu_{nj} t^j = \liminf_{j\to+\infty} |\nu_{nj}|^{-1/j}\ . \tag{3.21}$$

Let us relate these objects to the convergence radius τ_n in (3.15), to the solution $u \in C^\omega((-\mathfrak{T}, T), \mathcal{A})$ and to its squared H^n norm. We claim that

$$\tau_n \leqslant \theta_n \text{ and } \|u(t)\|_n^2 = \sum_{j=0}^{+\infty} \nu_{nj} t^j \text{ for } t \in (-\mathfrak{T}\wedge\theta_n, T\wedge\theta_n) \tag{3.22}$$

[3] Let us propose a proof of (3.20), based directly on the definition (3.19) of ν_{nj}. If $\mathcal{H}^-(u_0)$ has at least one element (S, a), from (3.6) and from the invariance of $\langle\ |\ \rangle_n$ under any transformation $\mathcal{E}_*(S, a)$ we obtain that, for each $\ell \in \{0, ..., j\}$, $\langle u_\ell | u_{j-\ell} \rangle_n = \langle (-1)^\ell \mathcal{E}_*(S, a) u_\ell | (-1)^{j-\ell} \mathcal{E}_*(S, a) u_{j-\ell} \rangle_n = (-1)^j \langle u_\ell | u_{j-\ell} \rangle_n$, whence $\nu_{nj} = (-1)^j \nu_{nj}$. If j is odd, this means $\nu_{nj} = 0$.

(with $-\mathcal{T} \wedge \theta_n$ the opposite of $\mathcal{T} \wedge \theta_n$). In fact: the expansion $u(t) = \sum_{j=0}^{+\infty} u_j t^j$, converging in H^n for $t \in (-\tau_n, \tau_n)$, implies $\tau_n \leqslant \theta_n$ and $\|u(t)\|_n^2 = \sum_{j=0}^{+\infty} \nu_{nj} t^j$ for $t \in (-\tau_n, \tau_n)$. Moreover the functions $t \in (-\mathcal{T}, T) \mapsto \|u(t)\|_n^2$ and $t \in (-\theta_n, \theta_n) \mapsto \sum_{j=0}^{+\infty} \nu_{nj} t^j$ are analytic and coincide on $(-\tau_n, \tau_n)$, so they coincide everywhere on the intersections of their domains, which is $(-\mathcal{T} \wedge \theta_n, T \wedge \theta_n)$. We now add to (3.22) a stronger claim:

$$\text{if } n > \frac{5}{2}, \quad \tau_n \leqslant \theta_n \leqslant \mathcal{T} \wedge T \text{ and } \|u(t)\|_n^2 = \sum_{j=0}^{+\infty} \nu_{nj} t^j \text{ for } t \in (-\theta_n, \theta_n). \tag{3.23}$$

Let us prove this claim, assuming for example that $\mathcal{T} \wedge T = T$. If it were $T < \theta_n$ we would infer $\lim_{t \to T^-} \|u(t)\|_n^2 = \lim_{t \to T^-} \sum_{j=0}^{+\infty} \nu_{nj} t^j = \sum_{j=0}^{+\infty} \nu_{nj} T^j < +\infty$ (the first equality would hold due to (3.22) and $T \wedge \theta_n = T$; the subsequent two relations would hold because T would be inside the convergence interval of the series). On the other hand, since $n > 5/2$, the conclusion that $\lim_{t \to T^-} \|u(t)\|_n^2$ exists finite would contradict (2.15).

4 Power series for the Euler equation in a paper of Behr, Nečas and Wu

In the paper [5] mentioned above, the authors considered the power series (3.1) for the Euler equation on $\mathbf{T}^3$, with an initial datum $u_0 \in \mathcal{A}$ given by

$$u_0(x) = \sum_{k = \pm a, \pm b, \pm c} u_{0k} e^{ik \bullet x} \ , \tag{4.1}$$

$$a := (1,1,0), \quad b := (1,0,1), \quad c := (0,1,1) \ ;$$

$$u_{0,\pm a} := (1,-1,0) \ , \quad u_{0,\pm b} := (1,0,-1) \ , \quad u_{0,\pm c} := (0,1,-1) \ .$$

Like u_0, all the subsequent terms u_j are Fourier polynomials with rational coefficients ([4]). Using rules equivalent to (3.2) (2.5), the terms u_j were determined in [5] by computer algebra, for $j = 1, 2, ..., 35$. Computations were done with Mathematica for $j = 1, ..., 10$, and with a C_{++} program for $j = 11, ..., 35$ (in the later case, approximating the rational coefficients with finite precision decimal numbers). After determining the u_j's, the authors fixed their attention on the partial sums

$$u^{(N)}(t) := \sum_{j=0}^{N} u_j t^j \ ,$$

[4]For a more precise statement on these cofficients see our discussion of the datum u_0 in the next section and, in particular, Eq. (5.9).

whose $N \to +\infty$ limit gives the solution $u(t)$ of the Euler Cauchy problem, for all t such that the series converges. The previously mentioned computation of the u_j's made available these partial sums for $N = 0, 1, ..., 35$; the authors of [5] computed the (squared) Sobolev norm

$$\|u^{(N)}(t)\|_3^2 = \sum_{k \in \mathbf{Z}^3} |k|^6 |u_k^{(N)}(t)|^2$$

for the above values of N, and several values of t. Their main results were the following:
(i) Setting $t = 0.32$, and analyzing the behavior of $\|u^{(N)}(0.32)\|_3$ for N from 0 to 35, the authors found evidence that $\|u^{(N)}(0.32)\|_3$ should approach a finite limit for $N \to +\infty$.
(ii) Setting $t = 0.35$, the authors observed a rapid growth of $\|u^{(N)}(0.35)\|_3$ for N ranging from 0 to 35, a fact suggesting that $\lim_{N \to +\infty} \|u^{(N)}(0.35)\|_3 = +\infty$.
(iii) A behavior as in (ii) was found to occur for slightly higher values of t (even though the authors suspected some rounding error to appear for $t > 0.35$).

The above results suggest that the series $\sum_{j=0}^{+\infty} u_j t^j$ has a finite convergence radius τ_3 in H^3, with $\tau_3 \in (0.32, 0.35)$.

Let us discuss this outcome from the viewpoint of the present paper, denoting with u the maximal $\mathcal{A}$-solution of the Cauchy problem with this datum and recalling that this coincides with the maximal H^3-solution. The datum u_0 possesses pseudo-symmetries (to be described in the next section); therefore, u has a time symmetric domain $(-T, T)$ (in [5] this fact was not explicitly declared, but probably regarded as self-evident). According to our Eq. (3.18), it is

$$\tau_3 \leqslant T \; ; \tag{4.2}$$

in principle, it could be $T = +\infty$. In spite of this, the authors of [5] spoke of a blow-up at τ_3.

In the next two sections we present our computations on the power series for the Behr-Nečas-Wu initial datum, with our interpretation of the results. Even though these calculations confirm the "experimental" outcomes (i)-(iii) of [5], we give evidence that the solution u of the Euler equation does not blow up close to τ_3; on the contrary, computing the power series for $\|u(t)\|_3^2$ up the available order we obtain strong evidence that such a power series has a convergence radius θ_3 such that $0.47 < \theta_3 < 0.50$, which implies for the time T of existence of u the bound $T \geqslant \theta_3 > 0.47$. By a subsequent analysis relying on the technique of the Padé approximants, we show that a blow-up of $u(t)$ might happen at a time larger than 0.48: more precisely, these computations give a somehow weak indication that T might be finite, with $0.56 < T < 0.73$.

5 Our approach to the power series of Behr, Nečas and Wu

Let us denote again with u_0 the datum (4.1) and consider its iterates u_j ($j = 1, 2, ...$), with the corresponding power series; like u_0, all the iterates u_j are Fourier polynomials with rational coefficients. Throughout the section, u is the maximal $\mathcal{A}$-solution of the Euler equation with datum u_0.

A closer analysis of the Behr-Nečas-Wu initial datum: symmetry properties. The symmetry group $\mathcal{H}(u_0)$ and the pseudo-symmetry space $\mathcal{H}^-(u_0)$ (Eqs. (2.34) (2.35)) can be explicitly computed. For the first one, we find

$$\mathcal{H}(u_0) = \{(\mathbf{1}, 0), (\mathbf{1}, \imath_2), (A, a_1), (A, a_2), (B, a_1), (B, a_2), \\ (C, c_1), (C, c_2), (D, c_1), (D, c_2), (E, 0), (E, \imath_2)\} \tag{5.1}$$

where $\mathbf{1}$ is the 3×3 identity matrix, and

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.2}$$

$$C := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad E := \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix};$$

furthermore, $\imath_2, a_1$, etc., are the following elements of $\mathbf{T}^3$:

$$\imath_2 := (\pi, \pi, \pi) \ , \quad a_1 := (0, 0, \pi) \ , \quad a_2 := (\pi, \pi, 0) \ , \\ c_1 := (\pi, 0, 0) \ , \quad c_2 := (0, \pi, \pi) \tag{5.3}$$

(of course, in the above π is short for π mod. $2\pi\mathbf{Z}$). Let us fix the attention on the reduced symmetry subgroup $\mathcal{H}_R(u_0) = \{\mathbf{1}, A, B, C, D, E\}$; it is readily checked that

$$A^3 = \mathbf{1}, B^2 = \mathbf{1}, (BA)^2 = \mathbf{1} \\ C = A^2, \quad D = AB \ , \quad E = A^2 B \ . \tag{5.4}$$

So, $\mathcal{H}_R(u_0)$ has two generators A, B; the first line in (5.4) gives a presentation of this group in terms of generators and relations, while the second line expresses the other elements in terms of A, B. Using Eq. (5.4), one recognizes a group isomorphism

$$\mathcal{H}_R(u_0) \simeq \mathbf{D}_3 \tag{5.5}$$

where the right-hand side indicates the dihedral group of order 3, formed by the symmetries of an equilateral triangle ([5]).

Now we consider the full group $\mathcal{H}(u_0)$ (with the product (2.22)). It is easy to check that

$$(A, a_1)^6 = (\mathbf{1}, 0) \ , \quad (B, a_1)^2 = (\mathbf{1}, 0) \ , ((B, a_1)(A, a_1))^2 = (\mathbf{1}, 0) \ , \tag{5.6}$$

$$(A, a_1)^2 = (C, c_2) \ , \quad (A, a_1)^3 = (\mathbf{1}, \imath_2) \ , \quad (A, a_1)^4 = (A, a_2) \ , \quad (A, a_1)^5 = (C, c_1) \ ,$$

$$(A, a_1)(B, a_1) = (D, c_2) \ , \quad (A, a_1)^2(B, a_1) = (E, \imath_2) \ , \quad (A, a_1)^3(B, a_1) = (B, a_2) \ ,$$

$$(A, a_1)^4(B, a_1) = (D, c_1) \ , \quad (A, a_1)^5(B, a_1) = (E, 0) \ .$$

So, $\mathcal{H}(u_0)$ has two generators (A, a_1) and (B, a_1); the first line in (5.6) gives a presentation of this group in terms of generators and relations, and the subsequent lines express the other elements in terms of the generators. One recognizes a group isomorphism

$$\mathcal{H}(u_0) \simeq \mathbf{D}_6 \tag{5.7}$$

where the right-hand side indicates the dihedral group of order 6, formed by the symmetries of a hexagon (see the previous footnote).

Let us pass to the pseudo-symmetry space $\mathcal{H}^-(u_0)$. One readily checks that this contains $(-\mathbf{1}, 0)$ (inducing the space reflection $\mathcal{E}(-\mathbf{1}, 0) : x \in \mathbf{T}^3 \mapsto -x$). From here and from the general result (2.40), one obtains

$$\mathcal{H}^-(u_0) = \mathcal{H}(u_0)(-\mathbf{1}, 0) = \{(-\mathbf{1}, 0), (-\mathbf{1}, \imath_2), (-A, a_1), (-A, a_2), \tag{5.8}$$

$$(-B, a_1), (-B, a_2), (-C, c_1), (-C, c_2), (-D, c_1), (-D, c_2), (-E, 0), (-E, \imath_2)\} \ ,$$

with $A, B, ...$ and $\imath_2, a_1, ...$ as in Eqs. (5.2) (5.3).

Some consequences of the previous symmetry results. (i) What we have stated in Section 3 for an arbitrary initial datum holds, in particular, for the present datum u_0: the symmetries or pseudo-symmetries of u_0 can be used to speed up the computation of the Fourier components of any iterate u_j. More precisely, if we know the Fourier component $u_{j,k}$ for some k, using Eqs. (3.10) (3.11) we readily obtain the components $u_{j,Sk}$ for all $S \in \mathcal{H}_R(u_0) \cup \mathcal{H}_R^-(u_0)$.

(ii) As already noted, the pseudo-symmetry space $\mathcal{H}^-(u_0)$ contains $(-\mathbf{1}, 0)$, corresponding to the space reflection. In terms of Fourier coefficients, the relation (3.11) with $(S, a) = (-\mathbf{1}, 0)$ takes the form $u_{j,-k} = (-1)^j u_{j,k}$ for $j = 0, 1, 2, ...$ and $k \in \mathbf{Z}^3$. On the other hand, any iterate u_j is a real vector field, thus $u_{j,-k} = \overline{u_{j,k}}$; in con-

[5] For any integer $n \in \{3, 4, ...\}$, one denotes with $\mathbf{D}_n$ the dihedral group of order n; this is formed by the orthogonal transformations of the Euclidean plane $\mathbf{R}^2$ into itself which preserve a regular polygon with n sides, centered at the origin. Denoting with id the identity map, with a the rotation of an angle $2\pi/n$ and with b the reflection about anyone of the n symmetry axes of the polygon, one finds that a, b are generators of $\mathbf{D}_n$ and fulfill the relations $a^n = id$, $b^2 = id$, $(ba)^2 = id$. The elements of $\mathbf{D}_n$ are $2n$, and coincide with $id, a, a^2, ..., a^{n-1}$, b, ab, a^2b, ..., $a^{n-1}b$.

clusion $\overline{u_{j,k}} = (-1)^j u_{j,k}$, which indicates that $u_{j,k}$ is real for j even, and imaginary for j odd. Taking into account that the coefficients $u_{j,k}$ are rational in any case, we conclude the following for each $k \in \mathbf{Z}^3$:

$$u_{j,k} \in \mathbf{Q}^3 \text{ for } j = 0, 2, 4, ... ; \qquad u_{j,k} \in i\mathbf{Q}^3 \text{ for } j = 1, 3, 5, ... \,. \tag{5.9}$$

(iii) In the sequel we are often interested in the partial sums $u^{(N)}(t) := \sum_{j=0}^{N} u_j t^j$ and in their norms $\|u^{(N)}(t)\|_n$, especially for $n = 3$. Since $\mathcal{H}^-(u_0) \neq \emptyset$, as in (3.9) we have $\|u_N(t)\|_n = \|u_N(-t)\|_n$.
(iv) Independently of any convergence consideration about the power series $\sum_{j=0}^{+\infty} u_j t^j$, the result $\mathcal{H}^-(u_0) \neq \emptyset$ also ensures that the (maximal $\mathcal{A}$-) solution u of the Euler equation with datum u_0 has a symmetric domain $(-T, T)$ (recall Eq. (2.32)).

Describing our computations. We have considered again the power series (3.1) for the datum u_0; to deal with this series we have written a program in Python, using the package gmpy [24] for fast arithmetics on rational numbers. This program implements Eq. (2.5) for $\mathcal{P}$ and the recursion rule (3.2); moreover, it takes into account the dihedral symmetries (and pseudosymmetries) of u_0 to speed up computations. The program has been run to compute the terms u_j for $j = 1, ..., 52$ ([6]). Calculations have been performed on a PC with an Intel Core i7 CPU 860 at 2.8GHz and an 8GB RAM. The CPU time for u_j has been, for example: 1 second for $j = 10$, one minute for $j = 20$, half an hour for $j = 30$, 7 hours for $j = 40$ and 85 hours for $j = 52$. Differently from [5], for *all* orders up to $j = 52$ the Fourier coefficients $u_{j,k}$ of u_j have been represented as elements of $\mathbf{Q}^3$ or $i\mathbf{Q}^3$; so, no rounding errors related to finite precision arithmetics have been introduced in the calculation of the power series.

From the u_j's one determines the squared norms $\|u_j\|_3^2 = \sum_{k \in \mathbf{Z}^3} |k|^6 |u_{j,k}|^2$, the partial sums $u^{(N)}(t) := \sum_{j=0}^{N} u_j t^j$ and their squared norms $\|u^{(N)}(t)\|_3^2 = \sum_{k \in \mathbf{Z}^3} |k|^6 |u_k^{(N)}(t)|^2$ ($N = 1, ..., 52$). Each $\|u_j\|_3^2$ is a rational number and $\|u^{(N)}(t)\|_3^2$ is a polynomial of order $2N$ in t, with rational coefficients, containing only even powers of t; furthermore, the coefficients of t^0 and t^{2N} in $\|u^{(N)}(t)\|_3^2$ are $\|u_0\|_3^2$ and $\|u_N\|_3^2$, respectively.

Our computations of the above norms, up to $j = 52$ or $N = 52$, have been done using the previously mentioned Python program. These calculations have been relatively quick: for example, the computation of $\|u^{(52)}(t)\|_3^2$ has required a CPU

[6]To test the reliability of this program, the calculation of some of the $u_j's$ has been checked in two independent ways. These checks have been done by means of other two programs, which implement Eqs. (2.5) (3.2) accepting as an initial datum u_0 any Fourier polynomial; these do not refer to any symmetry property of u_0. The first of these programs, written in Mathematica, has been used to compute the u_j's up to order $j = 13$; the second program, written in Python, has been used for a calculation up to $j = 43$.

time of about 3 hours. As first examples of our results, we report the following ones:

$$\|u_0\|_3^2 = 96, \quad \|u_1\|_3^2 = 6912, \quad \|u_2\|_3^2 = 45440, \tag{5.10}$$

$$\|u_3\|_3^2 = \frac{3695360}{9}, \quad \|u_4\|_3^2 = \frac{1366793248}{675}, \quad \|u_5\|_3^2 = \frac{2243123779689032}{186046875} .$$

$\|u_{52}\|_3^2$ is a ratio of integers where the numerator and the denominator have 19515 and 19463 digits, respectively. Table 1 reports $\|u_j\|_3^2$ for $j = 0, ..., 52$, in the 16 digits decimal representation.

Table 1. The squared norms $\|u_j\|_3^2$.

j	$\|u_j\|_3^2$	j	$\|u_j\|_3^2$
1	6912	27	$4.070323867244879 \times 10^{27}$
2	45440	28	$3.800202819232687 \times 10^{28}$
3	$4.105955555555556 \times 10^{5}$	29	$3.554589555246873 \times 10^{29}$
4	$2.024878885925926 \times 10^{6}$	30	$3.330557264153261 \times 10^{30}$
5	$1.205676676745595 \times 10^{7}$	31	$3.125627141295364 \times 10^{31}$
6	$8.452219877103332 \times 10^{7}$	32	$2.937654907691943 \times 10^{32}$
7	$6.152775603322622 \times 10^{8}$	33	$2.764771414352126 \times 10^{33}$
8	$4.791192836997696 \times 10^{9}$	34	$2.605347861791808 \times 10^{34}$
9	$3.628869598772102 \times 10^{10}$	35	$2.457968790658826 \times 10^{35}$
10	$2.825486371143428 \times 10^{11}$	36	$2.321406184470901 \times 10^{36}$
11	$2.228507964437443 \times 10^{12}$	37	$2.194593722846032 \times 10^{37}$
12	$1.821213808657725 \times 10^{13}$	38	$2.076602420620089 \times 10^{38}$
13	$1.539790191793044 \times 10^{14}$	39	$1.966618988613002 \times 10^{39}$
14	$1.341372343677860 \times 10^{15}$	40	$1.863927582086700 \times 10^{40}$
15	$1.190159209731028 \times 10^{16}$	41	$1.767894900465337 \times 10^{41}$
16	$1.066432595016119 \times 10^{17}$	42	$1.677958174980847 \times 10^{42}$
17	$9.598519025230687 \times 10^{17}$	43	$1.593615440091581 \times 10^{43}$
18	$8.662788463495777 \times 10^{18}$	44	$1.514417532673484 \times 10^{44}$
19	$7.840631870939454 \times 10^{19}$	45	$1.439961389630372 \times 10^{45}$
20	$7.122921654632158 \times 10^{20}$	46	$1.369884345517744 \times 10^{46}$
21	$6.499436510134908 \times 10^{21}$	47	$1.303859232337703 \times 10^{47}$
22	$5.957837347113741 \times 10^{22}$	48	$1.241590147950303 \times 10^{48}$
23	$5.485035371335649 \times 10^{23}$	49	$1.182808795435820 \times 10^{49}$
24	$5.068929708200902 \times 10^{24}$	50	$1.127271314453561 \times 10^{50}$
25	$4.699401376031744 \times 10^{25}$	51	$1.074755536205362 \times 10^{51}$
26	$4.368534165204974 \times 10^{26}$	52	$1.025058601409640 \times 10^{52}$

Let us pass to the squared norms $\|u^{(N)}(t)\|_3^2$. As an example, the result for $N = 5$ is

$$\|u^{(5)}(t)\|_3^2 = 96 + 6656\, t^2 + \frac{258304}{9}\, t^4 + \frac{104566912}{525}\, t^6 - \frac{9513575648}{70875}\, t^8 + \frac{2243123779689032}{186046875}\, t^{10}\ . \tag{5.11}$$

There is no room to report here the results obtained for all the other values of N, especially in the rational form for the coefficients. However, we can write some of them in the 16 digits precision; in particular,

$$\begin{aligned} \|u^{(52)}(t)\|_3^2 &\\ = 96 + 6656\, t^2 + 2.870044444444444 \times 10^4\, t^4 + 1.993359937918871 \times 10^5\, t^6 &\\ +1.058054454761424\times10^5\, t^8+1.781444415306641\times10^6\, t^{10}+2.740017914111055\times10^6\, t^{12} &\\ -7.321985472578865\times10^6\, t^{14}+4.183410651491110\times10^6\, t^{16}+1.457483700816015\times10^8\, t^{18} &\\ -1.768517246168822\times10^8\, t^{20}+4.196205149715839\times10^8\, t^{22}+3.648789154816725\times10^9\, t^{24} &\\ -2.178830191383206\times10^{10}\, t^{26}-1.394064522752687\times10^{10}\, t^{28}+2.954202883502504\times10^{11}\, t^{30} &\\ +1.283692616423054\times10^{11}\, t^{32}-4.543575106022102\times10^{12}\, t^{34}+4.789569007452901\times10^{12}\, t^{36} &\\ +2.830635227431622\times10^{13}\, t^{38}+4.470168139346678\times10^{13}\, t^{40}-6.910532995061547\times10^{14}\, t^{42} &\\ +1.457019276470951\times10^{14}\, t^{44}+9.053007124662626\times10^{15}\, t^{46}-8.939780851014422\times10^{15}\, t^{48} &\\ -1.019952729404346\times10^{17}\, t^{50}+1.137772938577812\times10^{17}\, t^{52}+1.644161010427522\times10^{18}\, t^{54} &\\ -4.571936581656874\times10^{18}\, t^{56}-3.140936865806385\times10^{19}\, t^{58}+2.408085513008218\times10^{21}\, t^{60} &\\ -1.107900217253947\times10^{23}\, t^{62}+4.186064092726056\times10^{24}\, t^{64}-1.674853723772203\times10^{26}\, t^{66} &\\ +6.911508987260593\times10^{27}\, t^{68}-2.698282390313396\times10^{29}\, t^{70}+9.951375797771149\times10^{30}\, t^{72} &\\ -3.558771163372845\times10^{32}\, t^{74}+1.232107326257251\times10^{34}\, t^{76}-4.045044388392564\times10^{35}\, t^{78} &\\ +1.242344004413423\times10^{37}\, t^{80}-3.561397641466941\times10^{38}\, t^{82}+9.520206481050174\times10^{39}\, t^{84} &\\ -2.357932432543021\times10^{41}\, t^{86}+5.354229494719748\times10^{42}\, t^{88}-1.103667607665446\times10^{44}\, t^{90} &\\ +2.052382635232918\times10^{45}\, t^{92}-3.436006560519912\times10^{46}\, t^{94}+5.184487278969682\times10^{47}\, t^{96} &\\ -7.072466985323957\times10^{48}\, t^{98}+8.759614973466463\times10^{49}\, t^{100}-9.896987665647683\times10^{50}\, t^{102} &\\ +1.025058601409640 \times 10^{52}\, t^{104}\ . & \end{aligned} \tag{5.12}$$

The rest of the paper reports a number of facts stemming from our computations, with the interpretation that we suggest for them.

Verification of the outcomes of [5] on $\|u^{(N)}(t)\|_3^2$. Our computations based on the systematic use of rational numbers have given essentially the same results as in [5] about $\|u^{(N)}(t)\|_3^2$ as a function of N, in the two cases $t = 0.32$ and $t = 0.35$. So, $\|u^{(N)}(0.32)\|_3^2$ seems to approach a limit value for large N, while $\|u^{(N)}(0.35)\|_3^2$ grows rapidly with N; our use of rational coefficients ensures that such a rapid growth is not due to cumulative rounding errors. In Figures 1-2, we report $\|u^{(N)}(t)\|_3^2$ as a function of $N \in \{0, ..., 52\}$, in the two cases $t = 0.32$ and $t = 0.35$; these figures are very similar to the ones at the bottom of pages 235 and 236 of [5], respectively (but comparison requires a rescaling, since the H^3 norm employed in [5] differs from ours by a constant factor).

We agree with [5] in interpreting these results as indications that the power series for this initial datum has a finite H^3-convergence radius τ_3, with $\tau_3 \in (0.32, 0.35)$.

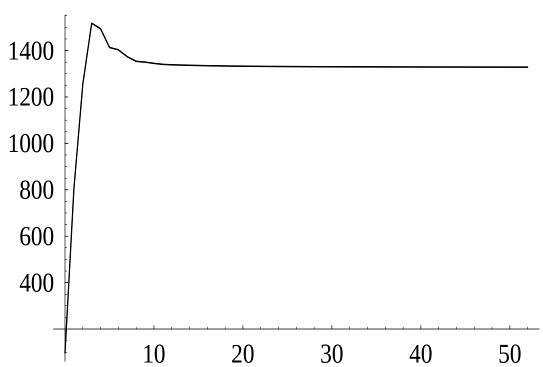

Figure 1. $\|u^{(N)}(0.32)\|_3^2$ as a function of $N \in \{0, 1, ..., 52\}$.

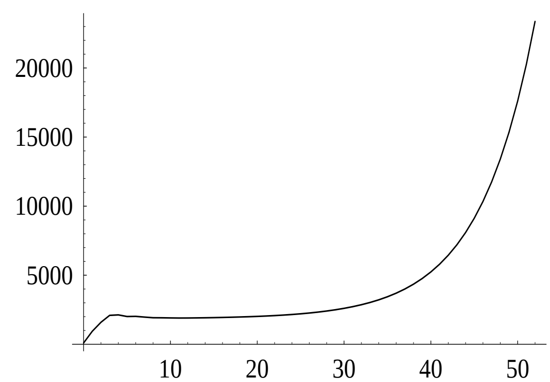

Figure 2. $\|u^{(N)}(0.35)\|_3^2$ as a function of $N \in \{0, 1, ..., 52\}$.

Further evidence on the H^3-convergence radius of the power series. This comes from the root test (3.16) for $n = 3$:

$$\tau_3 = \liminf_{j \to +\infty} \|u_j\|_3^{-1/j} \ . \tag{5.13}$$

Figure 3 represents $\|u_j\|_3^{-1/j}$ as a function of $j \in \{1, ..., 52\}$. For $j = 36, 38, ..., 52$ we have a very good interpolation

$$\|u_j\|_3^{-1/j} \simeq 0.32158 - \left(\frac{1.20125}{j}\right)^{1.38458} \ , \tag{5.14}$$

(obtained assuming for the interpolant the form $A - (B/j)^c$, and applying the least squares criterion). The right-hand side of (5.14) approximates $\|u_j\|_3^{-1/j}$ with a mean quadratic error $< 10^{-5}$ (averaging, as indicated, for $j = 36, 38, ..., 52$; if we average over the larger range $j = 16, 18, ..., 52$, the mean quadratic error is $< 10^{-4}$).

Assuming that (5.14) approximates $\|u_j\|_3^{-1/j}$ with a similar precision for arbitrarily large j, but keeping prudentially only two digits in our final estimate, we conclude with an estimate

$$0.32 < \tau_3 < 0.33 \ . \tag{5.15}$$

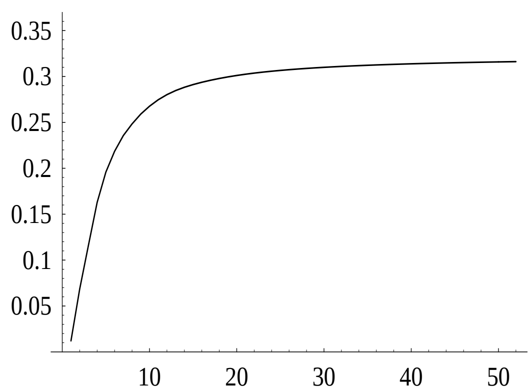

Figure 3. $\|u_j\|_3^{-1/j}$ as a function of $N \in \{1, ..., 52\}$.

Reminder estimates for the series expansion of $u(t)$ in H^3. Let $N \in \{0, 1, 2, ...\}$; of course

$$u(t) - u^{(N)}(t) = \sum_{j=N+1}^{+\infty} u_j t^j \qquad \text{for } t \in (-\tau_3, \tau_3) \; ; \tag{5.16}$$

this implies

$$\|u(t) - u^{(N)}(t)\|_3 \leqslant \sum_{j=N+1}^{+\infty} \|u_j\|_3 |t|^j \qquad \text{for } t \in (-\tau_3, \tau_3) \; . \tag{5.17}$$

To go on, we need a guess on the behavior of the norms $\|u_j\|_3$. To this purpose, let us consider the sequence

$$\mu_{3j} := 0.32^j \, \|u_j\|_3 \qquad (j = 0, 1, 2...) \; , \tag{5.18}$$

recalling that 0.32 is the lower bound for τ_3 in (5.15). From the norms available up to $j = 52$, we can check that (μ_{3j}) is decreasing while j ranges in $\{1, 3, ..., 52\}$; by extrapolation, let us assume that (μ_{3j}) is decreasing on the infinite set $\{1, 2, ...\}$. So, $\mu_{3j} \leqslant \mu_{3N}$ for integer $j \geqslant N \geqslant 1$, i.e.,

$$\|u_j\|_3 \leqslant \frac{\mu_{3N}}{0.32^j} \qquad \text{for } j \geqslant N \geqslant 1 \; . \tag{5.19}$$

For $t \in (-0.32, 0.32)$, inserting this inequality into (5.17) we get $\|u(t) - u^{(N)}(t)\|_3 \leqslant \mu_{3N} \sum_{j=N+1}^{+\infty} |t/0.32|^j = \mu_{3N} |t/0.32|^{N+1} \sum_{j=0}^{+\infty} |t/0.32|^j$, i.e.,

$$\|u(t) - u^{(N)}(t)\|_3 \leqslant \mu_{3N} \frac{|t/0.32|^{N+1}}{1 - |t/0.32|} \quad \text{for } t \in (-0.32, 0.32),\ N \in \{1, 2, 3, ...\}. \tag{5.20}$$

Of course, this is a conjecture based on the previous extrapolation. For the practical application of the reminder estimate (5.20), we mention that (rounding up from above)

$$\begin{aligned} &\mu_{3\,5} = 11.7,\ \mu_{3\,10} = 5.99,\ \mu_{3\,20} = 3.39, \\ &\mu_{3\,30} = 2.61,\ \mu_{3\,40} = 2.20,\ \mu_{3\,52} = 1.88\,. \end{aligned} \tag{5.21}$$

No blow-up at τ_3. After accumulating indications that the Taylor series for $u(t)$ has an H^3-convergence radius $\tau_3 \in (0.32, 0.33)$, in the rest of the section we will present evidence that $u(t)$ does *not* blow up at $t = \tau_3$.

The power series for $\|u(t)\|_3^2$; an indication that $u(t)$ exists up to $t = 0.47$ at least. The results (3.19) (3.20) with $n = 3$ give a formal series expansion

$$\|\sum_{j=0}^{+\infty} u_j t^j\|_3^2 = \sum_{j=0}^{+\infty} \nu_{3j} t^j, \quad \nu_{3j} := \sum_{\ell=0}^{j} \langle u_\ell | u_{j-\ell} \rangle_3 \in \mathbf{R}, \quad \nu_{3j} = 0 \quad \text{for } j \text{ odd} \; ; \tag{5.22}$$

the series $\sum_{j=0}^{+\infty} \nu_{3j} t^j$ has a convergence radius

$$\theta_3 = \liminf_{j \to +\infty} |\nu_{3j}|^{-1/j} \ . \tag{5.23}$$

Recalling that $(-T, T)$ is the domain of the solution u, we know (from (3.23)) that

$$\tau_3 \leqslant \theta_3 \leqslant T \ , \qquad \|u(t)\|_3^2 = \sum_{j=0}^{+\infty} \nu_{3j} t^j \qquad \text{for } t \in (-\theta_3, \theta_3). \tag{5.24}$$

In the sequel, for $N = 0, 1, 2, ...$ we also consider the partial sums

$$\nu_3^{(N)}(t) := \sum_{j=0}^{N} \nu_{3j} t^j \ . \tag{5.25}$$

Of course, $u^{(N)}(t) = \sum_{j=0}^{N} u_j t^j$ is such that $u(t) = u^{(N)}(t) + O(t^{N+1})$ for $t \to 0$; this implies $\|u(t)\|_3^2 = \|u^{(N)}(t)\|_3^2 + O(t^{N+1})$, whence

$$\nu_3^{(N)}(t) = \|u^{(N)}(t)\|_3^2 \Big|_{t^k \to 0 \text{ for } k > N} \ . \tag{5.26}$$

With this remark, the previuos computations of $\|u^{(N)}(t)\|_3^2$ up to $N = 52$ also give the partial sums $\nu_3^{(N)}(t)$ for $N = 1, ..., 52$ or, equivalently, the coefficients ν_{3j} for $j = 0, ..., 52$. For example,

$$\nu_{30} = 96, \quad \nu_{32} = 6656, \quad \nu_{34} = \frac{258304}{9}, \quad \nu_{36} = \frac{2825587712}{14175}, \tag{5.27}$$

$$\nu_{38} = \frac{52545219363488}{496621125}, \quad \nu_{3\,10} = \frac{1002532034046659735168576 8}{5627635784943046875} \ ;$$

$\nu_{3\,52}$ is a ratio of integers where the numerator and the denominator have 2610 and 2593 digits, respectively.

The 16-digits representation of the coefficients ν_{3j} for all $j \in \{0, ..., 52\}$ can be obtained from Eqs. (5.12) (5.26); more precisely,

$$\nu_{3j} = \text{coefficient of } t^j \text{ in (5.12), for } j = 0, ..., 52 \ . \tag{5.28}$$

From the above data, one can try to make predictions on the convergence radius θ_3 of the series $\sum_{j=0}^{+\infty} \nu_{3j} t^j$. In Figures 4-7 we report the partial sums $\nu_3^{(N)}(t)$ as functions of $N \in \{0, ..., 52\}$, in the four cases $t = 0.45, 0.50, 0.55, 0.60$. For $t = 0.45$, the function $N \mapsto \nu_3^{(N)}(t)$ seems to approach a limit value for large N. The situation is not clear for $t = 0.50$, due to the appearing of small oscillations; for $t = 0.55$ and $t = 0.60$, the oscillations of $N \mapsto \nu_3^{(N)}(t)$ are large and their amplitude increases with N. We regard these results as indicating that $\sum_{j=0}^{+\infty} \nu_{3j} t^j$ is convergent for $t \leqslant 0.45$ and not convergent for $t \geqslant 0.55$; in other words, for the convergence radius we have a conjectural estimate

$$0.45 < \theta_3 < 0.55 \ . \tag{5.29}$$

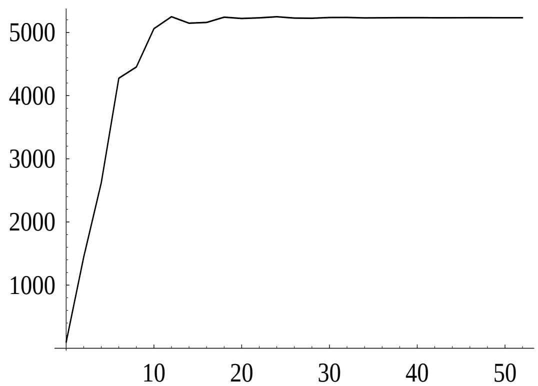

Figure 4. $\nu_3^{(N)}(0.45)$ as a function of $N \in \{0, 2, ..., 50, 52\}$.

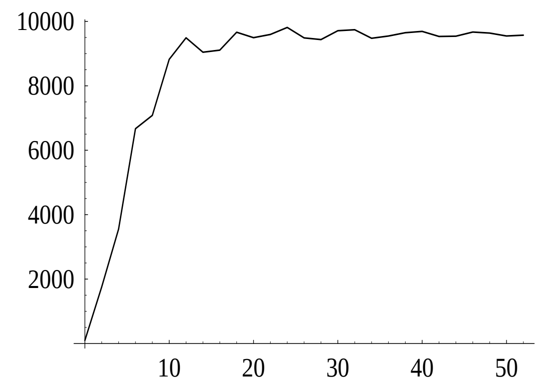

Figure 5. $\nu_3^{(N)}(0.50)$ as a function of $N \in \{0, 2, ..., 50, 52\}$.

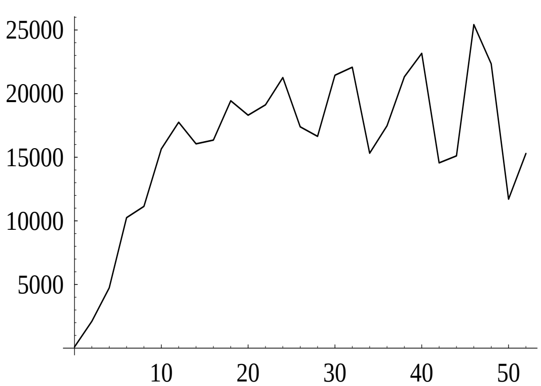

Figure 6. $\nu_3^{(N)}(0.55)$ as a function of $N \in \{0, 2, ..., 50, 52\}$.

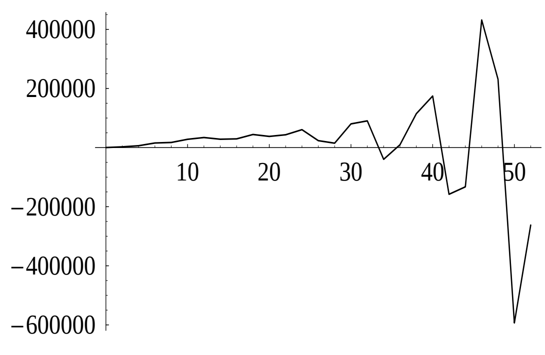

Figure 7. $\nu_3^{(N)}(0.60)$ as a function of $N \in \{0, 2, ..., 50, 52\}$.

Another way to estimate θ_3 comes from the root test (5.23). Figure 8 is a graph of $|\nu_{3j}|^{-1/j}$ as a function of $j \in \{2, 4, ..., 50, 52\}$. For $j = 36, 38, ..., 52$, there is a fairly good interpolation

$$|\nu_{3j}|^{-1/j} \simeq 0.484 - \left(\frac{8.48}{j}\right)^{2.19} \tag{5.30}$$

(obtained assuming for the interpolant the form $A - (B/j)^c$, and applying the least squares criterion); here, the right-hand side approximates $|\nu_{3j}|^{-1/j}$ with a mean quadratic error < 0.01 (let us repeat it, for j between 36 and 52). Assuming that the above interpolant behaves similarly for all larger (even) j, and considering $\theta_3 = \liminf_{j\to+\infty} |\nu_{3j}|^{-1/j}$ we are led to use 0.484 ± 0.01 as upper and lower bounds for it; rounding up to two digits we obtain the inequality

$$0.47 < \theta_3 < 0.50\ , \tag{5.31}$$

which is compatible with (5.29). Now, recalling that θ_3 is a lower bound on the time

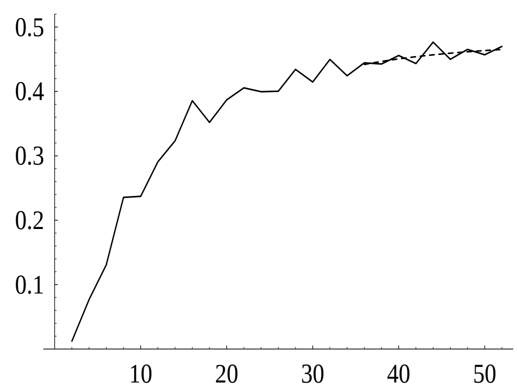

Figure 8. $|\nu_j|_3^{-1/j}$ as a function of $j \in \{2, 4, ..., 50, 52\}$. The dashed line is the graph of the interpolant in (5.30), for $j \in [36, 52]$.

of existence T of the solution u (see (5.24)), we are led to the final estimate

$$0.47 < T \leqslant +\infty\ . \tag{5.32}$$

In particular, as anticipated, we have indications that u does not blow up near the H^3-convergence radius τ_3.

6 Possible blow-up at larger times for the Behr-Nečas-Wu datum, via Padé approximants

A few words on Padé approximants. Let us be given an analytic function $f : I \to \mathbf{C}, t \mapsto f(t)$, with I a neighborhood of zero in $\mathbf{R}$ or $\mathbf{C}$. Let $p, q \in \{0, 1, 2, ...\}$; we recall that the Padé approximant of order (p, q) of f, if it exists, is the unique complex function $[p/q]_f \equiv [p/q]$ of the form

$$[p/q](t) = \frac{a_0 + a_1 t + ... + a_p t^p}{1 + b_1 t + ... b_q t^q} \ , \tag{6.1}$$

such that

$$f(t) = [p/q](t) + O(t^{p+q+1}) \quad \text{for } t \to 0 \ ; \tag{6.2}$$

the above condition determines the $p + q + 1$ unknown coefficients $a_0, ..., b_q$ as functions of the derivatives $f^{(j)}(0)$, $j = 0, ..., p + q$; the domain of $[p/q]$ is the largest subset of $\mathbf{C}$ where the above ratio is defined. The family of all approximants $[p/q]$ $(p, q = 0, 1, 2, ...)$ forms the so-called Padé table of f; the approximants with $p = q$ are called diagonal.

There are several results and conjectures about the convergence to f of the Padé approximants $[p/q]$ with p or p, q large. In particular, the so-called "Padé conjecture" (or "Baker-Gammel-Wills conjecture") states that, for a meromorphic function f on a disk of $\mathbf{C}$, there is a subsequence $[p_\ell/p_\ell]$ $(\ell = 1, 2, 3, ...)$ of diagonal Padé approximants that, for $\ell \to +\infty$, converges to f uniformly on each compact subset of the disk minus the poles of f. This conjecture has been proved for special classes of meromorphic functions (see [1] [21] [22] and references therein).

It is found experimentally that the Padé approximants of large order (and, in particular, the diagonal approximants $[p/p]$) work as well for many non meromorphic functions, describing accurately their behavior even close to non polar singularities.

Padé approximants for $\|\boldsymbol{u}(t)\|_3^2$, and possible evidence for a blow-up. The previous considerations can be applied (for suitable n) to the function $f(t) := \|u(t)\|_n^2$, where u is the solution of the Euler equation with a given datum u_0.

One can ascribe to a number of works the idea of using the Padé approximants for such a function; as in the Introduction, we mention [8] [12] [15] [20] (and some references therein). As already remarked, these papers have considered initial data u_0 different from the one of Behr-Nečas-Wu (e.g., the Taylor-Green vortex); furthermore, they have generally considered the Sobolev norm of order $n = 1$.

Here we are focusing on the (maximal $\mathcal{A}$-) solution u for the Behr-Nečas-Wu datum; from now on, $[p/q]$ stands for the Padé approximants of the analytic function

$$t \mapsto f(t) := \|u(t)\|_3^2 \ . \tag{6.3}$$

We conjecture that, for certain large p, $[p/p]$ approximates well the function $t \mapsto \|u(t)\|_3^2$ (and even its analytic continuation to the complex plane). From the previous paragraphs, we have the derivatives $f^{(j)}(0) = j!\,\nu_{3j}$ for $j = 0, ..., 52$; this information suffices to determine all the Padé approximants $[p/q]$ for $p+q \leqslant 52$ and, in particular, all the diagonal approximants $[p/p]$ for $p = 0, 1, ..., 26$.

It turns out that the diagonal approximants $[p/p]$ exist in the cases of even order $p = 0, 2, ..., 26$, while they do not exist in the odd cases $p = 1, 3, ..., 25$ (the reason being, essentially, that the power series for $f(t)$ about zero contains only even powers of t). Let us consider, for example, the approximant [12/12]. Its numerator and denominator are polynomials with rational coefficients, too large to be written explicitly; however, we can use the 16-digits approximation for the coefficients and write

$$[12/12](t) = \frac{N_{12}(t)}{D_{12}(t)} \ , \tag{6.4}$$

$$\begin{aligned} N_{12}(t) &:= 96 + 6.680481407149543 \times 10^3\, t^2 + 3.08095009988031 \times 10^4\, t^4 \\ &\quad + 2.3462351635051233 \times 10^5\, t^6 + 2.407391215430808 \times 10^5\, t^8 \\ &\quad + 2.5575522886490226 \times 10^6\, t^{10} + 3.094974424148063 \times 10^6\, t^{12} \ , \end{aligned}$$

$$\begin{aligned} D_{12}(t) &:= 1 + 0.255014657807743\, t^2 + 4.288322833232482\, t^4 - 5.985294148961588\, t^6 \\ &\quad + 8.973150435320479\, t^8 + 66.29326162173366\, t^{10} - 612.1107629833056\, t^{12} \ . \end{aligned}$$

The poles of [12/12], which are the zeros of D_{12}, are simple and occur at the points

$$\begin{aligned} & t = \pm 0.294020 \pm 0.464361\, i \quad (|t| = 0.549617) \ ; \\ & t = \pm 0.511609 \pm 0.301416\, i \quad (|t| = 0.593797) \ ; \\ & t = \pm 0.606004\, i \ , \qquad t = \pm 0.626199 \end{aligned} \tag{6.5}$$

(here and in the sequel, $\pm$ means that we can choose independently the signs for the real and imaginary part, e.g., $+$ for the real and $-$ for the imaginary part). So, the singularities of minimum modulus of the approximant [12/12] are at anyone of the points $T_\circ = \pm 0.294020 \pm 0.464361\, i$, such that $|T_\circ| = 0.549617$; furthermore, the real singularities closest to the origin are at anyone of the points $T_* = \pm 0.626199$.

We have performed a similar analysis for all the approximants $[p/p]$, with $p = 14, 16, ..., 26$; the results are summarized in Table 2.

Table 2. Poles of the Padé approximants $[p/p](t)$ to $\|u(t)\|_3^2$.
$T_\circ :=$ pole closest to the origin (with modulus $|T_\circ|$);
$T_* :=$ real (or almost real) pole closest to the origin.

$[p/p]$	$T_\circ$	$\|T_\circ\|$	T_*
$[12/12]$	$\pm 0.294020 \pm 0.464361\, i$	0.549617	± 0.626199
$[14/14]$	$\pm 0.281333 \pm 0.445002\, i$	0.526474	± 0.656185
$[16/16]$	$\pm 0.283300 \pm 0.446498\, i$	0.528790	± 0.661087
$[18/18]$	$\pm 0.283081 \pm 0.445859\, i,$	0.528134	± 0.660118
$[20/20]$	$\pm 0.345307 \pm 0.348713\, i$	0.490752	$\pm 0.621387 \pm 0.047708\, i$
$[22/22]$	$\pm 0.350239 \pm 0.350695\, i$	0.495635	± 0.541967
$[24/24]$	$\pm 0.349063 \pm 0.350777\, i$	0.494863	$\pm 0.609804 \pm 0.0383530\, i$
$[26/26]$	$\pm 0.0714399 \pm 0.508700\, i$	0.513692	± 0.816133

Let us point out some features of the Padé approximants Table 2, with their possible implications:
(i) For all the approximants $[p/p]$ in the table, the poles of minimum modulus occur at points $T_\circ$ with $|T_\circ| \simeq 0.5$. There is not a clear trend of $|T_\circ|$ as a function of p, so we limit ourself to consider the mean of $|T_\circ|$ for $p = 12, ..., 26$ which is $\langle |T_\circ| \rangle = 0.515995$, with a mean quadratic error $\Delta T_\circ < 0.02$. On the other hand, for a holomorphic function, the convergence radius of the power series centered at zero is the modulus of the singularity closest to the origin. So, assuming that the above $[p/p]$ describe approximately the singularities of $f(t) = \|u(t)\|_3^2$, we can derive from these approximants an estimate of a convergence radius θ_3 for the power series of $f(t)$. More precisely, assuming $|T_\circ| - \Delta T_\circ < \theta_3 < |T_\circ| + \Delta T_\circ$ and rounding up to two digits, we obtain from the above Padé approximants an estimate

$$0.49 < \theta_3 < 0.54 \ ; \tag{6.6}$$

this is compatible with the estimate on θ_3 obtained in Section 5 by other means (see Eq. (5.31) and the discussion before it).
(ii) The $[p/p]$ approximants of Table 2 have real poles (symmetric with respect to the origin), with the exceptions of $[20/20]$ and $[24, 24]$ which, however, possess "almost real" poles, close to the real axis ([7]). In the table, we have denoted with T_* the real (or almost real) singularities closest to the origin. The mean of $|T_*|$ for for $p = 12, ..., 26$ is $\langle |T_*| \rangle = 0.649489$, with a mean quadratic error $\Delta T_* < 0.08$ (however, there are large deviations from the mean in the special cases $p = 22$ and $p = 26$).

[7] The occurring of almost real singularities has also been pointed out in [12] [20] while analyzing the Padé approximants for $\|u(t)\|_1^2$, with initial conditions u_0 different from the Behr-Nečas-Wu datum.

The above results on the singularities T_* somehow suggest that $f(t) = \|u(t)\|_3^2$ could diverge for $t \to T^-$ (and $t \to (-T)^+$), for a suitable T; if we assume for T the upper and lower bounds $|T_*| \pm \Delta T_*$, rounding up to two digits we get

$$0.56 < T < 0.73 \ . \tag{6.7}$$

If such a conjectured divergence of $f(t)$ actually occurred, the solution u of the Euler equation with the Behr-Nečas-Wu datum would blow up at T (and $-T$); admittedly, the indications for such a blow up are very weak.

D-log Padè approximants for $\|u(t)\|_3^2$. As well known, the D-log Padé approximants of a function $t \mapsto f(t)$ are the Padé approximants for the logarithmic derivative $\dot{f}/f$ ($\dot{} := d/dt$). These approximants are generally regarded as more suitable for describing the behavior of f close to singularities, even of non polar type. In particular, the presence of a singularity at a point T_*, say real, and a behavior of the type $[p/p]_{\dot{f}/f} \sim \lambda_*/(T_* - t)$ for $t \to T_*^-$ is regarded as an indication that $f(t) \sim \text{const}/(T-t)^\lambda$ for real $t \to T^-$, where $T \simeq T_*$ and $\lambda \simeq \lambda_*$ [1].

We have attempted an analysis of the function $f(t) := \|u(t)\|_3^2$ via the approximants $[p/p]_{\dot{f}/f}$, with odd $p \leqslant 25$ ([8]); the results are very unstable with respect to the order, and ultimately not sufficient to get any indication of blow-up. ([9])

[8] Our function has the form $f(t) = F(t^2)$; in such a case, for odd p, the D-log approximant of f of order (p, p) is (up to a factor $2t$) the D-log approximant of the function $s \mapsto F(s)$ of order $(p/2 - 1/2, p/2 - 1/2)$. On the contrary, for even p, the (p, p) D-log approximant of f cannot be interpreted in terms of F.

[9] Here is a more precise description of the computational outcomes. The D-log approximants of order (p, p) for $p = 17, 19, 21$ have real singularities at points $T_* \simeq 0.72$ and are such that $[p/p]_{\dot{f}/f} \sim \lambda_*/(T_* - t)$ for $t \mapsto T_*^-$, with $\lambda_* \simeq 2.6$; so, for $\|u(t)\|_3 = \sqrt{f(t)}$ we have a conjecture $\|u(t)\|_3 \sim \text{const.}/(T-t)^\alpha$ with $T \simeq 0.72$ and $\alpha \simeq \lambda_*/2 \simeq 1.3$. This value of α agrees with the Beale-Kato-Majda bound $\alpha \geqslant 1$ in the event of blow-up (see Eq. (2.16)); it agrees as well with the (conjectural) bound $\alpha \geqslant 6/5$, obtained extrapolating from $\mathbf{R}^3$ to $\mathbf{T}^3$ the estimate (2.17).

On the contrary, the D-log approximant of order $(23, 23)$ for f has no real (nor almost real) singularity. Finally, at the order $(25, 25)$ there is a real singularity for $T_* \simeq 0.52$, and $[25/25]_{\dot{f}/f} \sim \lambda_*/(T_* - t)$ for $t \mapsto T_*^-$, with $-0.002 < \lambda_* < 0.002$ (there are numerical difficulties in a more precise determination of λ_*). Returning to $\|u(t)\|_3 = \sqrt{f(t)}$, the $[25, 25]$ Padé would suggest $\|u(t)\|_3 \sim \text{const.}/(T-t)^\alpha$ with $T \simeq 0.52$ and $-0.001 \lesssim \alpha \lesssim 0.001$. This statement is an absurdity even in the case $0 < \alpha \lesssim 0.001$, since it contradicts the Beale-Kato-Majda bound (2.16) $\alpha \geqslant 1$.

7 Conclusions

The previous results about the Behr-Nečas-Wu datum u_0 support our statements in the Introduction, i.e.:
(a) The power series for u_0 has an H^3 convergence radius τ_3 such that $0.32 < \tau_3 < 0.33$ (see Eq. (5.15)).
(b) There is no blow-up at time τ_3 and the (maximal $\mathcal{A}$-) solution u of the Euler Cauchy problem exists, at least, up to a time θ_3 (the convergence radius for the series expansion of $\|u(t)\|_3^2$), for which we have from (5.31) the estimate $\theta_3 > 0.47$.
(c) The Padé approximants for $\|u(t)\|_3^2$ in Table 2 give weak indications that u might blow up at a time T, with $0.56 < T < 0.73$ (see Eq. (6.7)).
We think that the evidence given in this paper is rather strong for (a)(b). As for (c), doubts on the blow-up conjecture arise not only from the rather erratic behavior of the real singularities in the computed Padé approximants; in fact there are more general reasons, recalled at the end of the Introduction, suggesting caution in deriving blow-up results from the Padé approximants.

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References

[1] G.A. Baker, P. Graves-Morris, "Padé approximants", 2nd Edition, Encyclopedia of Mathematics and its Applications **59**, Cambridge University Press, Cambridge (1996).

[2] M.S. Baouendi, C. Goulaouic, *Sharp estimates for analytic pseudodifferential operators and application to Cauchy problems*, J. Differential Equations **48** (1983) 241-268.

[3] C. Bardos, E. S. Titi, *Euler equations for incompressible ideal fluids*, Russian Math. Surveys **62** (2007) 409-451.

[4] J. T.Beale, T. Kato, A. Majda, *Remarks on the breakdown of smooth solutions for the 3D Euler equations*, Commun. Math. Phys. **94** (1984) 61-66.

[5] E. Behr, J. Nečas, H. Wu, *On blow-up of solution for Euler equations*, ESAIM: M2AN **35** (2001) 229-238.

[6] N. Bourbaki, "Éléments de Mathématique. Variétés différentielles et analytiques, Fascicule de résultats", Hermann, Paris (1971).

[7] M.E. Brachet, D. Meiron, S. Orszag, B. Nickel, R. Morf, U. Frisch, *Small scale structure of the Taylor-Green vortex*, J. Fluid Mech. **130** (1983) 411-452.

[8] M.E. Brachet, D. Meiron, S. Orszag, B. Nickel, R. Morf, U. Frisch, *The Taylor-Green vortex and fully developed turbulence*, J. Statist. Phys. **34** (1984) 1049-1063.

[9] M.E. Brachet, M. Meneguzzi, A. Vincent, H. Politano, P.L. Sulem, *Numerical evidence of smooth self-similar dynamics and possibility of subsequent collapse for three-dimensional ideal flows*, Phys. Fluids A **4** (1992) 2845-2854.

[10] T. Chen, N. Pavlović, *A lower bound on blowup rates for the 3D incompressible Euler equation and a single exponential Beale-Kato-Majda estimate*, arXiv:1107.0435v1 [math.AP] (2011).

[11] S.I. Chernyshenko, P. Constantin, J.C. Robinson, E.S. Titi, *A posteriori regularity of the three-dimensional NavierStokes equations from numerical computations*, J. Math. Phys. **48** (2007) 065204.

[12] U. Frisch, *Fully developed turbulence and singularities*, in: G. Ioss, R.H.G. Helleman, R. Stora (Eds.), "Chaotic Behavior of Deterministic Systems": LesHouches, session XXXVI, North-Holland, Amsterdam (1983) 665-704.

[13] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations*, in "Spectral theory and differential equations", Proceedings of the Dundee Symposium, Lecture Notes in Mathematics **448** (1975) 23-70.

[14] S. Kida, *Three-dimensional periodic flows with high-symmetry*, J. Phys. Soc. Japan **54** (1985) 2132-2140.

[15] R.H. Morf, S.A. Orszag, U. Frisch, *Spontaneous singularity in three-dimensional inviscid, incompressible flow*, Phys. Rev. Letts. **44** (1980) 572-574.

[16] M. Morimoto, "Analytic functionals on the sphere", Translations of Mathematical Monographs, **178**, AMS (1998).

[17] C. Morosi, L. Pizzocchero, *On approximate solutions of semilinear evolution equations II. Generalizations, and applications to Navier-Stokes equations*, Rev. Math. Phys. **20** (2008) 625-706.

[18] C. Morosi, L. Pizzocchero, *An H^1 setting for the Navier-Stokes equations: Quantitative estimates*, Nonlinear Anal. **74** (2011) 2398-2414.

[19] C. Morosi, L. Pizzocchero, *On approximate solutions of the incompressible Euler and Navier-Stokes equations*, Nonlinear Anal. **75** (2012) 2209-2235.

[20] R. B. Pelz, *Extended series analysis of full octahedral flow: numerical evidence for hydrodynamic blowup*, Fluid Dynamics Research **33** (2003) 207-221.

[21] H. Stahl, *The convergence of diagonal Padé approximants and the Padé conjecture*, J. Comput. Appl. Math. **86** (1997) 287-296.

[22] S.P. Suetin, *Padé approximants and efficient analytic continuation of a power series*, Russian Math. Surveys **57** (2002) 43-141.

[23] F. Treves, "Topological vector spaces, distributions and kernels", Academic Press, New York (1967).

[24] GMPY Collaboration, "Multiprecision arithmetic for Python", http://code.google.com/p/gmpy. This software is a wrapper for GMP Multiple Precision Arithmetic Library, see http://gmplib.org.